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GENERALIZED CONTINUED FRACTION ALGORITHM FOR THE INDEX 3 SUBLATTICE

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Abstract. Motivated by an algorithm to generate all Pythagorean triples, Romik introduced a dynamical system on the unit circle, which corresponds the continued fraction algorithm on the index-2 sublattice. Cha et al. extended Romik's work to other ellipses and spheres and developed a dynamical system generating all Eisenstein triples. In this article, we review the dynamical systems by Romik and by Cha et al. and find connections to the continued fraction algorithms.

1. Introduction

We have continued fraction expansion of a real number as

$$[a_1, a_2, a_3, \dots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}, \qquad a_i \in \mathbb{N} \quad \text{for} \ \ i \geq 1.$$

The continued fraction can be studied using dynamical systems of the Gauss map or the Farey map (cf. [5]). The Farey map $T_F : [0,1] \to [0,1]$ is given by

$$T_F([a_1, a_2, \dots]) := \begin{cases} [a_1 - 1, a_2, \dots] & \text{if} \ \ a_1 \geq 2, \\ [a_2, a_3, \dots] & \text{if} \ \ a_1 = 1 \end{cases}$$

and the continued fraction expansion is obtained by the acceleration of the symbolic coding of the Farey map. We extend the relation between the Farey map and the continued fraction to more general cases. Let

$$S := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \ \text{ and } \ x, y \geq 0\}$$

be the unit circle in the first quadrant. A rational point $\mathbf{z} = (\frac{a}{c}, \frac{b}{c}) \in S$ is denoted by a primitive Pythagorean triple (a, b, c), satisfying $a^2 + b^2 = c^2$, where a, b, c

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are nonnegative integers. The Berggren tree [1] gives an algorithm to generate all primitive Pythagorean triples and Romik [11] defined a dynamical system on the quarter circle S by the algorithm of the Berggren tree. Let

$$[a_1, \varepsilon_1 \!:\! a_2, \varepsilon_2 \!:\! a_3, \dots]_2 := \cfrac{1}{2a_1 + \cfrac{\varepsilon_1}{2a_2 + \cfrac{\varepsilon_2}{2a_3 + \ddots}}},$$

where $a_i \in \mathbb{N}$ and $\epsilon_i \in \{-1, 1\}$ for $i \geq 1$, be the even continued fraction expansion (cf. [14]). By the stereographic projection, Romik's dynamical system $T_R : [0,1] \to [0,1]$ on the unit interval to itself satisfies

$$(1) \qquad T_R([a_1, \varepsilon_1 \!:\! a_2, \varepsilon_2 \!:\! a_3, \dots]_2) = \begin{cases} [a_1 - 1, \varepsilon_1 \!:\! a_2, \varepsilon_2 \!:\! a_3, \dots]_2 & \text{if } a_1 \geq 2, \\ [a_2, \varepsilon_2 \!:\! a_3, \varepsilon_3 \!:\! a_4, \dots]_2 & \text{if } a_1 = 1. \end{cases}$$

Therefore, the even continued fraction is an acceleration of the symbolic coding of the Romik map T_R. For the detailed relation between the Romik map T_R and the even continued fraction map, consult [6].

We denote the index-p sublattice of $\mathbb{Z}^2$ and the set of corresponding rational numbers by

$$\Lambda_1^{(p)} := \{(n,m) \in \mathbb{Z}^2 \,|\, n = m \pmod{p}\}, \quad \mathbb{Q}_1^{(p)} := \left\{ \frac{n}{m} \in \mathbb{Q} \,|\, (n,m) \in \Lambda_1^{(p)} \right\}.$$

Then the Romik map T_R preserves $\mathbb{Q}_1^{(2)}$ or the index-2 sublattice $\Lambda_1^{(2)}$. Indeed, all convergents of the even continued fraction expansion are in $\mathbb{Q} \setminus \mathbb{Q}_1^{(2)}$. See [16] for more discussions on the convergents of the even continued fraction and [9] for the geometric meaning of the even continued fraction. It is related with the spectrum on 2-minimal form by the sublattice of index 2 was studied by Asmus Schmidt [12] (see [13] for $p = 3$) and by Vulakh (see [10]). See also [8,3,7] for the discussion with the 2-minimal form and the continued fraction on the circle.

Let

$$E := \{(x,y) \in \mathbb{R}^2 \mid x^2 + xy + y^2 = 1 \text{ and } x, y \geq 0\}$$

be the ellipse on the first quadrant. In a similar way with the Berggren tree, Cha et al [4] introduced an algorithm to generate all primitive Eisenstein triples (a,b,c) satisfying $a^2 + ab + b^2 = c^2$ and using it and the stereographic projection map $f_E : E \to [0,1]$, they defined a dynamical system $T_C : [0,1] \to [0,1]$.

In this article, we consider the Cha map T_C and the related continued fraction algorithm. While the Romik map T_R preserves $\mathbb{Q}_1^{(2)}$, the Cha map T_C preserves $\mathbb{Q}_1^{(3)}$

or the index-3 sublattice $\Lambda_1^{(3)}$. We will find the invariant density of T_C (Theorem 3.3). Using symbolic coding of the map T_C, we introduce a new continued fraction algorithm fixing the the index-3 sublattice $\Lambda_1^{(3)}$. Then we will show that all principal convergents p_n/q_n belongs to $\mathbb{Q} \setminus \mathbb{Q}_1^{(3)}$ (Theorem 3.4). In Section 2, we review the Farey map, the Romik map and the even continued fraction algorithm. Then the main results are given in Section 3. Finally, geometric interpretation of the new continued fraction is given in Section 4.

2. Farey Map, Romik Map and the Sublattice of Index 2

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z})$. Then M acts on the upper half plane hyperbolic surface $\mathbb{H} = \{x + yi \in \mathbb{C} \,|\, y > 0\}$ as well as its boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ as

$$M \cdot z = \begin{cases} \dfrac{az+b}{cz+d} & \text{if } \det(M) = 1, \\ \dfrac{a\bar{z}+b}{c\bar{z}+d} & \text{if } \det(M) = -1. \end{cases}$$

We write the continued fraction expansion with matrices as

$$[a_1, a_2, \dots] = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_n \end{pmatrix} \cdot [a_{n+1}, a_{n+2}, \dots]$$

$$= \overbrace{L_1 \cdots L_1}^{a_1-1} L_2 \overbrace{L_1 \cdots L_1}^{a_2-1} L_2 \cdots \overbrace{L_1 \cdots L_1}^{a_n-1} L_2 \cdot [a_{n+1}, a_{n+2}, \dots], \tag{2}$$

where

$$L_1 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \qquad L_2 := \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The Farey map, introduced to find intermediate convergents by S. Ito [5] is defined by

$$T_F(t) = \begin{cases} \dfrac{t}{1-t} = (L_1)^{-1} \cdot t & \text{if } 0 \le t \le \frac{1}{2}, \\ \dfrac{1-t}{t} = (L_2)^{-1} \cdot t & \text{if } \frac{1}{2} < t \le 1. \end{cases}$$

See [15] for the geometric meaning of the Farey map.

For each $t \in [0, 1]$, we have the symbolic coding of the Farey map by the sequence $(d_j^F)_{j=1}^{\infty}$ given by

$$d_j^F(t) := \begin{cases} 1 & \text{if } T_F^{j-1}(t) \in [0, \frac{1}{2}], \\ 2 & \text{if } T_F^{j-1}(t) \in (\frac{1}{2}, 1], \end{cases}$$

where $T_F^{j-1}(t) = (\overbrace{T_F \circ \cdots \circ T_F}^{j-1})(t)$. We write

$$[\![d_1^F, d_2^F, d_3^F, \dots]\!]_F := t.$$

Note that for all $k \geq 1$

$$[\![d_1, d_2, \dots]\!]_F = L_{d_1} L_{d_2} \cdots L_{d_k} \cdot [\![d_{k+1}, d_{k+2}, \dots]\!]_F.$$

Using (2), we deduce that for each $[\![d_1, d_2, \dots]\!]_F = [a_1, a_2, \dots,]$, we have

$$d_k = \begin{cases} 1 & \text{if } \ a_1 + \cdots + a_{n-1} < k < a_1 + \cdots + a_n \ \text{ for some } n, \\ 2 & \text{if } \ k = a_1 + \cdots + a_n \ \text{ for some } n. \end{cases}$$

For $k = a_1 + \cdots + a_n + \ell$ with $0 \leq \ell \leq a_{n+1} - 1$, we have

$$\begin{aligned} L_{d_1} \cdots L_{d_k} \cdot 0 &= \overbrace{L_1 \cdots L_1}^{a_1-1} L_2 \overbrace{L_1 \cdots L_1}^{a_2-1} L_2 \cdots \overbrace{L_1 \cdots L_1}^{a_n-1} L_2 L_1^\ell \cdot 0 \\ &= \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} L_1^\ell \cdot 0 = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \cdot 0 = \frac{p_n}{q_n}, \end{aligned}$$

where p_n/q_n is the n-th convergent of the continued fraction expansion. Therefore, the set of convergents of the continued fraction of $t = [\![d_1, d_2, \dots]\!]_F$ is equal to

$$\{ L_{d_1} L_{d_2} \cdots L_{d_k} \cdot 0 \,|\, k \geq 0 \}.$$

Now we consider continued fraction algorithms on the circle S, where the rational points are given by primitive Pythagorean triples. Berggren [1] developed a method to generated all primitive Pythagorean triple; if (a, b, c) is a primitive Pythagorean triple, there exists a unique sequence $d_1, \dots, d_k$ of digits $d_j \in \{1, 2, 3\}$ such that

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \tilde{M}_{d_1} \cdots \tilde{M}_{d_k} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \tilde{M}_{d_1} \cdots \tilde{M}_{d_k} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

where $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ are defined to be

$$\tilde{M}_1 := \begin{pmatrix} -1 & 2 & 2 \\ -2 & 1 & 2 \\ -2 & 2 & 3 \end{pmatrix}, \quad \tilde{M}_2 := \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 3 \end{pmatrix}, \quad \tilde{M}_3 := \begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & 2 \\ 2 & -2 & 3 \end{pmatrix}.$$

Note that the multiplications by $\tilde{M}_i$, $i = 1, 2, 3$, do not change the parity, thus a primitive Pythagorean triple (a, b, c) satisfies either

$$(a, b, c) \equiv (1, 0, 1) \pmod 2 \quad \text{or} \quad (a, b, c) \equiv (0, 1, 1) \pmod 2.$$

Romik [11] defined a dynamical system T_R on the quarter circle S by the action of the inverse of matrices $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$. By the stereographic projection map $f_S : S \to [0,1]$ such that

$$f_S(x,y) = \frac{y}{1+x}, \qquad f_S^{-1}(t) = \left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}\right),$$

the matrices actions of $\tilde{M}_1, \tilde{M}_2, \tilde{M}_3$ become to actions of

$$M_1 := \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \qquad M_2 := \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}. \qquad M_3 := \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

Therefore, Romik's dynamical system $T_R : [0,1] \to [0,1]$ is given by

$$T_R(t) := \begin{cases} M_1^{-1} \cdot t = \dfrac{t}{1-2t} & \text{if } \ 0 \le t \le \frac{1}{3}, \\ M_2^{-1} \cdot t = \dfrac{1-2t}{t} & \text{if } \ \frac{1}{3} < t < \frac{1}{2}, \\ M_3^{-1} \cdot t = \dfrac{2t-1}{t} & \text{if } \ \frac{1}{2} \le t \le 1. \end{cases}$$

As before, for each $t \in [0,1]$, we define the symbolic coding of the Romik map

$$[\![d_1^R, d_2^R, d_3^R, \ldots]\!]_R := t$$

by the sequence $(d_j^R)_{j=1}^{\infty}$ of

$$d_j^R(t) := \begin{cases} 1 & \text{if } \ T_R^{j-1}(t) \in [0, \frac{1}{3}], \\ 2 & \text{if } \ T_R^{j-1}(t) \in (\frac{1}{3}, \frac{1}{2}), \\ 3 & \text{if } \ T_R^{j-1}(t) \in [\frac{1}{2}, 1]. \end{cases}$$

Let $p/q \in \mathbb{Q} \cap [0,1]$ be a reduced form and (a,b,c) be the primitive Pythagorean triple satisfying

$$f_S^{-1}\left(\frac{p}{q}\right) = \left(\frac{q^2-p^2}{q^2+p^2}, \frac{2pq}{q^2+p^2}\right) = \left(\frac{a}{c}, \frac{b}{c}\right).$$

Then

$$(a,b,c) = \begin{cases} (q^2-p^2, 2pq, q^2+p^2) \equiv (1,0,1) \pmod 2 & \text{if } \ p/q \in \mathbb{Q} \setminus \mathbb{Q}_1^{(2)}, \\ \left(\dfrac{q^2-p^2}{2}, \dfrac{2pq}{2}, \dfrac{q^2+p^2}{2}\right) \equiv (0,1,1) \pmod 2 & \text{if } \ p/q \in \mathbb{Q}_1^{(2)}. \end{cases}$$

Therefore, for a primitive Pythagorean triple (a,b,c) we have

$$f_S\left(\frac{a}{c}, \frac{b}{c}\right) = \frac{b}{a+c} \in \begin{cases} \mathbb{Q}_1^{(2)} & \text{if } \ (a,b,c) \equiv (0,1,1) \pmod 2, \\ \mathbb{Q} \setminus \mathbb{Q}_1^{(2)} & \text{if } \ (a,b,c) \equiv (1,0,1) \pmod 2 \end{cases}$$

Using even continued fraction expansion, we have

$$[a_1, \varepsilon_1 : a_2, \varepsilon_2 : a_3, \dots]_2 = \begin{pmatrix} 0 & 1 \\ \epsilon_1 & 2a_1 \end{pmatrix} \cdot [a_2, \varepsilon_2 : a_3, \dots]_2$$
$$= \begin{cases} M_1^{a_1-1} M_2 \cdot [a_2, \varepsilon_2 : a_3, \dots]_2, & \text{if } \varepsilon_1 = 1, \\ M_1^{a_1-1} M_3 \cdot [a_2, \varepsilon_2 : a_3, \dots]_2, & \text{if } \varepsilon_1 = -1. \end{cases}$$

Therefore, we deduce (1) and if $[a_1, \varepsilon_1 : a_2, \varepsilon_2 : a_3, \dots]_2 = [\![d_1, d_2, \dots]\!]_R$, then

$$d_k = \begin{cases} 1 & \text{if } a_1 + \cdots + a_{n-1} < k < a_1 + \cdots + a_n \text{ for some } n, \\ 2 & \text{if } k = a_1 + \cdots + a_n \text{ for some } n \text{ and } \varepsilon_n = 1, \\ 3 & \text{if } k = a_1 + \cdots + a_n \text{ for some } n \text{ and } \varepsilon_n = -1. \end{cases}$$

For $k = a_1 + \cdots + a_n + \ell$ with $0 \le \ell \le a_{n+1} - 1$, we have

$$M_{d_1} \cdots M_{d_k} \cdot 0 = \overbrace{M_1 \cdots M_1}^{a_1 - 1} M_{d(\epsilon_1)} \overbrace{M_1 \cdots M_1}^{a_2 - 1} M_{d(\epsilon_2)} \cdots \overbrace{M_1 \cdots M_1}^{a_n - 1} M_{d(\epsilon_n)} M_1^{\ell} \cdot 0$$
$$= \begin{pmatrix} \varepsilon_n p_{n-1} & p_n \\ \varepsilon_n q_{n-1} & q_n \end{pmatrix} L_1^{\ell} \cdot 0 = \begin{pmatrix} \varepsilon_n p_{n-1} & p_n \\ \varepsilon_n q_{n-1} & q_n \end{pmatrix} \cdot 0 = \frac{p_n}{q_n},$$

where $d(1) = 2$, $d(-1) = 3$ and p_n/q_n is the n-th convergent of the even continued fraction expansion. It is straightforward to check that

$$M_i \left(\Lambda_1^{(2)} \right) = \Lambda_1^{(2)} \quad \text{and} \quad M_i^{-1} \left(\Lambda_1^{(2)} \right) = \Lambda_1^{(2)} \qquad \text{for} \quad i = 1, 2, 3.$$

Therefore, the Romik map T_R preserves two sets of the rational numbers

$$T_R \left(\mathbb{Q}_1^{(2)} \right) = \mathbb{Q}_1^{(2)}, \qquad T_R \left(\mathbb{Q} \setminus \mathbb{Q}_1^{(2)} \right) = \mathbb{Q} \setminus \mathbb{Q}_1^{(2)}.$$

Note that for $t = [\![d_1, d_2, \dots]\!]_R$ the set of convergents p_n/q_n of the even continued fraction is equal to

$$\{ M_{d_1} \cdots M_{d_k} \cdot 0 \,|\, k \ge 0 \} \subset \mathbb{Q} \setminus \mathbb{Q}_1^{(2)}.$$

3. MAIN RESULTS

Set

$$N_1 := \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, N_2 := \begin{pmatrix} 0 & 1 \\ 1 & 3 \end{pmatrix}, N_3 := \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}, N_4 := \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}, N_5 := \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}.$$

Using the stereographic projection map $f_E : E \to [0, 1]$ satisfying

$$f_E(x, y) = \frac{y}{1 + x}, \qquad f_E^{-1}(t) = \left(\frac{1 - t^2}{1 + t + t^2}, \frac{t(2 + t)}{1 + t + t^2} \right)$$

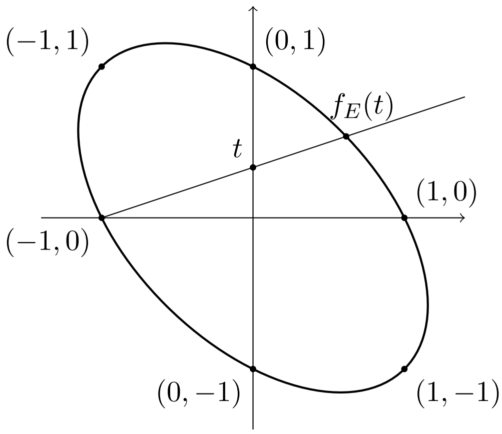

Figure 1. The ellipse E and the projection map f_E

(see Figure 1), Cha et al [4] defined the Cha map $T_C : [0,1] \to [0,1]$ by

$$T_C(t) := \begin{cases} N_1^{-1} \cdot t = \dfrac{t}{1-3t} & \text{if } 0 \le t \le \frac{1}{4}, \\ N_2^{-1} \cdot t = \dfrac{1-3t}{t} & \text{if } \frac{1}{4} \le t \le \frac{1}{3}, \\ N_3^{-1} \cdot t = \dfrac{3t-1}{1-2t} & \text{if } \frac{1}{3} \le t \le \frac{2}{5}, \\ N_4^{-1} \cdot t = \dfrac{1-2t}{3t-1} & \text{if } \frac{2}{5} \le t \le \frac{1}{2}, \\ N_5^{-1} \cdot t = \dfrac{2t-1}{t} & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \tag{3}$$

(see also [2]). Like the Romik map T_R which preserves modulo 2 parities, the Cha map T_C preserves the modulo 3 parities.

Proposition 3.1. *We have*

$$T_C\left(\mathbb{Q}_1^{(3)}\right) = \mathbb{Q}_1^{(3)}, \qquad T_C\left(\mathbb{Q} \setminus \mathbb{Q}_1^{(3)}\right) = \mathbb{Q} \setminus \mathbb{Q}_1^{(3)}.$$

Proof. Since

$$\begin{aligned} n \equiv m \pmod 3 \quad &\text{if and only if} \quad n \equiv 3n + m \pmod 3, \\ n \equiv m \pmod 3 \quad &\text{if and only if} \quad m \equiv n + 3m \pmod 3, \\ n \equiv m \pmod 3 \quad &\text{if and only if} \quad n + m \equiv 2n + 3m \pmod 3, \\ n \equiv m \pmod 3 \quad &\text{if and only if} \quad n + m \equiv 3n + 2m \pmod 3, \\ n \equiv m \pmod 3 \quad &\text{if and only if} \quad m \equiv -n + 2m \pmod 3, \end{aligned}$$

we have

$$N_i^{-1} \cdot \mathbb{Q}_1^{(3)} = N_i \cdot \mathbb{Q}_1^{(3)} = \mathbb{Q}_1^{(3)} \quad \text{for all } \ i = 1, 2, 3, 4, 5. \qquad \square$$

Similar to the Pythagorean triple, there are two classes of the Eisenstein triples.

Proposition 3.2. *Let (a, b, c) be a primitive Eisenstein triple. Then we have*

$$f_E\left(\frac{a}{c}, \frac{b}{c}\right) = \frac{b}{a+c} \in \begin{cases} \mathbb{Q}_1^{(3)}, & \text{if } \ a + 1 \equiv b \pmod 3, \\ \mathbb{Q} \setminus \mathbb{Q}_1^{(3)}, & \text{if } \ a \equiv b + 1 \pmod 3. \end{cases}$$

Proof. Let $p/q \in \mathbb{Q} \cap [0, 1]$ be a reduced form. Then

$$f_E^{-1}\left(\frac{p}{q}\right) = \left(\frac{q^2 - p^2}{q^2 + qp + p^2}, \frac{p(2q + p)}{q^2 + qp + p^2}\right).$$

Suppose that r is a common prime factor of $q^2 - p^2$, $p(2q + p)$, $q^2 + qp + p^2$. Then we have that $r \mid (q - p)(q + p)$ and that $r \mid p(2q + p)$. Since p and q are coprime, we have $r \nmid p$, $r \mid (q - p)$ and $r \mid (2q + p)$. Thus, we have $r \mid 3q$ and $r = 3$. Moreover, if 3 is the common factor of $q^2 - p^2$, $p(2q + p)$, $q^2 + qp + p^2$, then we have

$$q \equiv p \not\equiv 0 \pmod 3.$$

Also, it is immediate to check that 3 is a common factor of $q^2 - p^2$, $p(2q + p)$, $q^2 + qp + p^2$ if $p \equiv q \pmod 3$.

Let $p/q \in \mathbb{Q} \cap [0, 1]$ be a reduced form and (a, b, c) be the primitive Einstein triple satisfying

$$f_E^{-1}\left(\frac{p}{q}\right) = \left(\frac{a}{c}, \frac{b}{c}\right).$$

Then we deduced that

$$(a, b, c) = \begin{cases} (q^2 - p^2, p(2q + p), q^2 + qp + p^2) & \text{if } \ p/q \in \mathbb{Q} \setminus \mathbb{Q}_1^{(3)}, \\ \left(\dfrac{q^2 - p^2}{3}, \dfrac{p(2q + p)}{3}, \dfrac{q^2 + qp + p^2}{3}\right) & \text{if } \ p/q \in \mathbb{Q}_1^{(3)}. \end{cases}$$

Moreover, for $p/q \in \mathbb{Q} \setminus \mathbb{Q}_1^{(3)}$, we have

$$(a, b, c) \equiv \begin{cases} (1, 0, 1) \pmod 3 & \text{if } \ p \equiv 0 \pmod 3, \\ (-1, 1, 1) \pmod 3 & \text{if } \ q \equiv 0 \pmod 3, \\ (0, -1, 1) \pmod 3 & \text{if } \ p + q \equiv 0 \pmod 3. \end{cases} \tag{4}$$

On the other hand, for coprime p, q with $p \equiv q \not\equiv 0 \pmod 3$, we have

$$a + 1 = (q + p)\frac{(q - p)}{3} + 1 \equiv 2p\frac{(q - p)}{3} + p^2 \equiv p\,\frac{2q + p}{3} = b \pmod 3.$$

Therefore, for $p/q \in \mathbb{Q}_1^{(3)}$ we have

$$(a, b, c) \equiv (0, 1, 1) \quad \text{or} \quad (-1, 0, 1) \quad \text{or} \quad (1, -1, 1) \pmod 3. \tag{5}$$

By (4) and (5), we complete the proof. □

Theorem 3.3. *The Cha map T_C on $[0, 1]$ has invariant density* $\dfrac{dt}{t(1-t)}$.

Proof. We need to check that for each $t \in (0, 1)$

$$\rho(t) = \sum_{x \in T^{-1}(\{t\})} \frac{\rho(x)}{|T'(x)|}$$

for the invariant density $\rho(t) = \frac{1}{t(1-t)}$. Given $t \in (0, 1)$, we have five inverse images of T_C

$$x_1 = \frac{t}{3t+1}, \quad x_2 = \frac{1}{t+3}, \quad x_3 = \frac{t+1}{2t+3}, \quad x_4 = \frac{t+1}{3t+2}, \quad x_5 = \frac{1}{2-t}.$$

Then we have

$$\begin{aligned}
&\sum_{x \in T_C^{-1}(\{t\})} \frac{\rho(x)}{|T'(x)|} \\
&= \frac{\rho(x_1)}{|T'(x_1)|} + \frac{\rho(x_2)}{|T'(x_2)|} + \frac{\rho(x_3)}{|T'(x_3)|} + \frac{\rho(x_4)}{|T'(x_4)|} + \frac{\rho(x_5)}{|T'(x_5)|} \\
&= \frac{(1-3x_1)^2}{x_1(1-x_1)} + \frac{(x_2)^2}{x_2(1-x_2)} + \frac{(1-2x_3)^2}{x_3(1-x_3)} + \frac{(3x_4-1)^2}{x_4(1-x_4)} + \frac{(x_5)^2}{x_5(1-x_5)} \\
&= \frac{(1-\frac{3t}{3t+1})^2}{\frac{t}{3t+1}(1-\frac{t}{3t+1})} + \frac{\frac{1}{t+3}}{1-\frac{1}{t+3}} + \frac{(1-2\frac{t+1}{2t+3})^2}{\frac{t+1}{2t+3}(1-\frac{t+1}{2t+3})} + \frac{(3\frac{t+1}{3t+2}-1)^2}{\frac{t+1}{3t+2}(1-\frac{t+1}{3t+2})} + \frac{\frac{1}{2-t}}{1-\frac{1}{2-t}} \\
&= \frac{1}{t(1+2t)} + \frac{1}{t+2} + \frac{1}{(t+1)(t+2)} + \frac{1}{(t+1)(2t+1)} + \frac{1}{1-t} = \frac{1}{t(1-t)} = \rho(t),
\end{aligned}$$

which complete the proof. □

For each $t \in [0, 1]$, we define the symbolic coding of the Cha map T_C by

$$[\![d_1^C, d_2^C, d_3^C, \ldots]\!]_C := t,$$

where

$$d_j^C(t) := \begin{cases} 1 & \text{if } T_R^{j-1}(t) \in [0, \frac{1}{4}], \\ 2 & \text{if } T_R^{j-1}(t) \in (\frac{1}{4}, \frac{1}{3}], \\ 3 & \text{if } T_R^{j-1}(t) \in (\frac{1}{3}, \frac{2}{5}), \\ 4 & \text{if } T_R^{j-1}(t) \in [\frac{2}{5}, \frac{1}{2}), \\ 5 & \text{if } T_R^{j-1}(t) \in [\frac{1}{2}, 1]. \end{cases}$$

Then for each $k \geq 1$

$$[\![d_1, d_2, d_3, \dots]\!]_C = N_{d_1} N_{d_2} \cdots N_{d_k} \cdot [\![d_{k+1}, d_{k+2}, \dots]\!]_C.$$

Let

$$\alpha(d) := \begin{cases} 0 & \text{if } d = 2 \text{ or } 3, \\ 1 & \text{if } d = 4 \text{ or } 5, \end{cases} \qquad \beta(d) := \begin{cases} 0 & \text{if } d = 2 \text{ or } 5, \\ 1 & \text{if } d = 3 \text{ or } 4. \end{cases}$$

Then for $d = 2, 3, 4, 5$

$$N_1^{a-1} N_d = \begin{pmatrix} 0 & 1 \\ (-1)^d & 3a - \alpha(d) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \beta(d) & 1 \end{pmatrix}.$$

For $a_i \in \mathbb{N}$ and $\alpha_i, \beta_i \in \{0, 1\}$ for $i \geq 1$, define

$$[a_1, (\alpha_1, \beta_1) \colon a_2, (\alpha_2, \beta_2) \dots]_3 := \cfrac{1}{3a_1 - \alpha_1 + \cfrac{(-1)^{\alpha_1 + \beta_1}}{\beta_1 + 3a_2 - \alpha_2 + \cfrac{(-1)^{\alpha_2 + \beta_2}}{\beta_2 + 3a_3 - \alpha_3 + \ddots}}}.$$

Then

$$[a_1, (\alpha_1, \beta_1) \colon a_2, (\alpha_2, \beta_2) \dots]_3 = \overbrace{N_1 \cdots N_1}^{a_1 - 1} N_{d(\alpha_1, \beta_1)} \cdot [a_2, (\alpha_2, \beta_2) \colon a_3, (\alpha_3, \beta_3) \dots]_3,$$

where

$$d(\alpha, \beta) := \begin{cases} 2 & \text{if } \alpha = 0, \ \beta = 0, \\ 3 & \text{if } \alpha = 0, \ \beta = 1, \\ 4 & \text{if } \alpha = 1, \ \beta = 1, \\ 5 & \text{if } \alpha = 1, \ \beta = 0. \end{cases}$$

Therefore, if $[a_1, (\alpha_1, \beta_1) \colon a_2, (\alpha_2, \beta_2) \dots]_3 = [\![d_1, d_2, \dots]\!]_C$, then

$$d_k = \begin{cases} 1 & \text{if } a_1 + \cdots + a_{n-1} < k < a_1 + \cdots + a_n \text{ for some } n, \\ d(\alpha_n, \beta_n) & \text{if } k = a_1 + \cdots + a_n \text{ for some } n. \end{cases}$$

For $k = a_1 + \cdots + a_n + \ell$ with $0 \leq \ell \leq a_{n+1} - 1$, we have

$$N_{d_1} \cdots N_{d_k} \cdot 0 = \overbrace{N_1 \cdots N_1}^{a_1 - 1} N_{d(\alpha_1, \beta_1)} \cdots \overbrace{N_1 \cdots N_1}^{a_n - 1} N_{d(\alpha_n, \beta_n)} N_1^{\ell} \cdot 0$$

$$= \overbrace{N_1 \cdots N_1}^{a_1-1} N_{d(\alpha_1,\beta_1)} \cdots \overbrace{N_1 \cdots N_1}^{a_{n-1}-1} N_{d(\alpha_{n-1},\beta_{n-1})} \cdot \frac{1}{3a_n + \beta_{n-1} - \alpha_n}$$

$$= \cfrac{1}{3a_1 - \alpha_1 + \cfrac{(-1)^{\alpha_1+\beta_1}}{\beta_1 + 3a_2 - \alpha_2 + \cfrac{(-1)^{\alpha_2+\beta_2}}{\ddots + \cfrac{(-1)^{\alpha_{n-1}+\beta_{n-1}}}{\beta_{n-1} + 3a_n - \alpha_n}}}}.$$

Therefore, $N_{d_1} \cdots N_{d_k} \cdot 0$ is the n-th convergent of the continued fraction expansion of $[\![a_1, (\alpha_1, \beta_1) : a_2, (\alpha_2, \beta_2), a_3, \ldots]\!]_3$. Combined with Proposition 3.1, we have the following theorem.

Theorem 3.4. *All convergents p_n/q_n belongs to $\mathbb{Q} \setminus \mathbb{Q}_1^{(3)}$.*

4. Matrix Action on the Hyperbolic Space

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{Z})$. Then M acts on the upper half plane hyperbolic surface $\mathbb{H} = \{x + yi \in \mathbb{C} \,|\, y > 0\}$ as well as its boundary $\partial\mathbb{H} = \mathbb{R} \cup \{\infty\}$ by

$$M \cdot z = \begin{cases} \dfrac{az+b}{cz+d}, & \text{if } \det(M) = 1, \\ \dfrac{a\bar{z}+b}{c\bar{z}+d}, & \text{if } \det(M) = -1. \end{cases}$$

The fundamental domain of $\mathrm{PGL}_2(\mathbb{Z})$ is given in Figure 2. Note that the fundamental domain of $\mathrm{PGL}_2(\mathbb{Z})$ is the half of the fundamental domain of $\mathrm{PSL}_2(\mathbb{Z})$.

There is an natural homomorphism

$$\varphi : \mathrm{PGL}_2(\mathbb{Z}) \to \mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z}).$$

Let

$$\mathbf{G}(3) = \left\{ M \in \mathrm{PGL}_2(\mathbb{Z}) \;\middle|\; \varphi(M) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then $\mathbf{G}(3)$ is a index 24-subgroup of $\mathrm{PGL}_2(\mathbb{Z})$ since there are 24 elements of $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z})$. See Figure 2 for the fundamental domain of $\mathbf{G}(3)$. Let

$$\mathbf{S}_1(3) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \right\},$$

be a subgroup of $\mathrm{PGL}_2(\mathbb{Z}/3\mathbb{Z})$. Let

$$\mathbf{G}_1(3) = \{M \in \mathrm{PGL}_2(\mathbb{Z}) \mid \varphi(M) \in \mathbf{S}_1(3)\}.$$

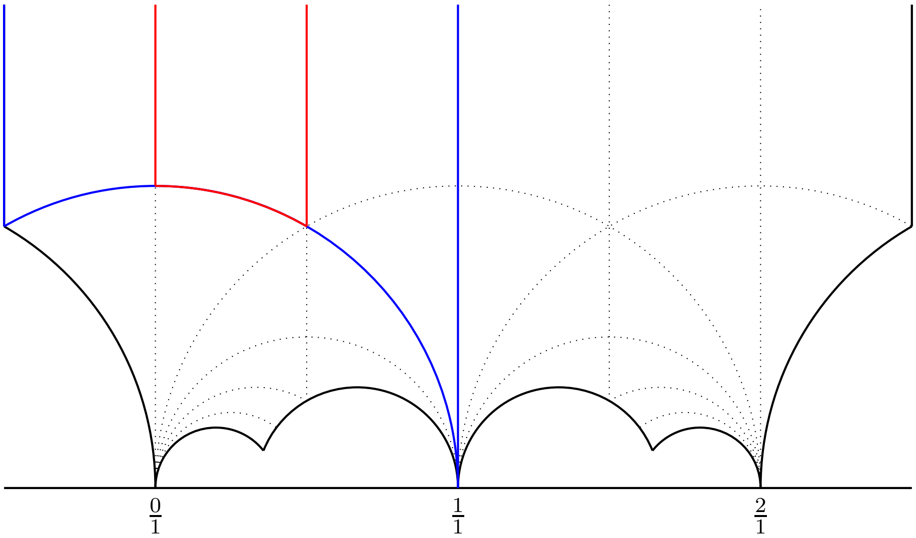

Figure 2. Fundamental domains of $\mathrm{PGL}_2(\mathbb{Z})$ (red), $\mathbf{G}_1(3)$ (blue) and $\mathbf{G}(3)$ (black)

Then $\mathbf{G}_1(3)$ is an index-4 subgroup of $\mathrm{PGL}_2(\mathbb{Z})$ fixing $\mathbb{Q}_1^{(3)}$ or the index-3 sublattice $\Lambda_1^{(3)}$. See Figure 2 for the fundamental domain of $\mathbf{G}_1(3)$, which is 4 copies of the fundamental domain of $\mathrm{PGL}_2(\mathbb{Z})$. There are two cusps in the fundamental domain of $\mathbf{G}_1(3)$. The cusp ∞ corresponds to $\mathbb{Q} \setminus \mathbb{Q}_1^{(3)}$ and the cusp 1 corresponds to $\mathbb{Q}_1^{(3)}$.

Since $N_1, \dots, N_5 \in \mathbf{G}_1(3)$, the expansion along the Cha map gives Diophantine approximation on $\mathbf{G}_1(3)$. For $t = [\![d_1, d_2, d_3, \dots]\!]_C$, we have two convergents

$$r_k := N_{d_1} N_{d_2} \cdots N_{d_k} \cdot 0 \quad \text{and} \quad s_k := N_{d_1} N_{d_2} \cdots N_{d_k} \cdot 1.$$

We check that

$$r_k \in \mathbb{Q} \setminus \mathbb{Q}_1^{(3)}, \; s_k \in \mathbb{Q}_1^{(3)} \quad \text{and} \quad t = \lim_{k\to\infty} r_k = \lim_{k\to\infty} s_k.$$

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