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# INNOVATION FIXED POINT OF COMPARISON AND WEAKLY CONTRACTIVE FUNCTION

Ghorban Khalilzadeh Ranjbar

Abstract. We prove some new fixed point theorems in class of generalized metric with using comparison functions and almost generalized weakly contractive mappings. Finally, we give an example to illustrate our main results.

## 1. Introduction and Preliminaries

The concept of generalized metric space is similar to that of metric space. However, it is very difficult to treat this concept because generalized metric space does not necessarily have the topology, which is compatible with [11]. So, this concept is very interesting to researchers. In this paper $T$-$b$-generalized metric space are introduced. Some fixed point results dealing with rational type contraction and almost generalized weakly contractive are obtained. The following definition was given by Branciari [2].

**Definition 1.1** ([2]). Let $X$ be a non-empty set and $d : X \times X \to \mathbb{R}^+$ be a mapping such that, for all $x, y \in X$ and for all distinct points $u, v \in X$ each distance from $u$ and $v$. of them different from $x$ and $y$, we have

($r_1$) $d(x, y) = 0$ if and only if $x = y$;
($r_2$) $d(x, y) = d(y, x)$;
($r_3$) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then $(X, d)$ is called a *generalized metric space* or shortly (g.m.s).

**Example 1.2** ([2]). Let $A = \{0, 2\}$, $B = \{\frac{1}{n} : n \in \mathbb{N}\}$ and $X = A \cup B$. Define $d : X^2 \to [0, \infty)$ as follows.

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$$d(x,y)=\begin{cases}0, & x=y,\\ 1, & x\neq y \text{ and } \{x,y\}\subset A \text{ or } \{x,y\}\subset B,\\ y, & x\in A, y\in B,\\ x, & x\in B, y\in A.\end{cases}$$

Then $(X,d)$ is a g.m.s.

**Lemma 1.3** ([11])**.** *Let $(X,d)$ be a g.m.s, and let $\{x_n\}$ be a Cauchy sequence in $X$ such that $x_n\neq x_m$, whenever $n\neq m$. Then $\{x_n\}$ converges to at most one point.*

### 1.1. Main results

**Theorem 1.4.** *Let $(Y,\preceq,d)$ a complete ordered g.m.s. Let $g:Y\to Y$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0\in Y$ with $x_0\preceq gx_0$. Suppose that*

$$d(gx,gy)\leq kM(x,y) \tag{1.1}$$

*for all elements $x,y\in Y$, where $0\leq k<1$ is real number and*

$$M(x,y)=\max\left\{d(x,y),\frac{d(x,gx)d(y,gy)}{1+d(gx,gy)},\frac{d(x,gx)d(y,gy)}{1+d(x,y)},\frac{d(x,gx)d(x,gy)}{1+d(x,gy)+d(y,gx)}\right\}. \tag{1.2}$$

*If $g$ is continuous, then $g$ has a fixed point. Moreover, the set of fixed points of $g$ is well ordered if and only if $g$ has one and one fixed point.*

*Proof.* Fix given $y_0$, put $y_n=g^ny_0$. If $y_n=y_{n+1}$ for some $n$, then $y_n=gy_n$. Thus $y_n$ is a fixed point of $g$. Therefore, we will assume that $y_n\neq y_{n+1}$ for all $n$. Since $y_0\preceq gy_0$ and $g$ is an increasing function, we obtain by induction that

$$y_0\preceq gy_0\preceq g^2y_0\preceq\cdots\preceq g^ny_0\preceq g^{n+1}y_0\preceq\cdots.$$

STEP 1. We will show that $\lim_{n\to\infty}d(y_n,y_{n+1})=0$. Since $y_n\preceq y_{n+1}$ for each $n$, by (1.1), we have

$$d(y_n,y_{n+1})=d(gy_{n-1},gy_n)\leq kM(y_{n-1},y_n), \tag{1.3}$$

where

$$
\begin{aligned}
&M\left(y_{n-1}, y_{n}\right) \\
&\quad=\max \left\{d\left(y_{n-1}, y_{n}\right), \frac{d\left(y_{n-1}, g y_{n-1}\right) d\left(y_{n}, g y_{n}\right)}{1+d\left(g y_{n-1}, y_{n}\right)}, \frac{d\left(y_{n-1}, g y_{n-1}\right) d\left(y_{n}, g y_{n}\right)}{1+d\left(y_{n-1}, y_{n}\right)},\right. \\
&\qquad \left.\frac{d\left(y_{n-1}, g y_{n-1}\right) d\left(y_{n-1}, g y_{n}\right)}{1+d\left(y_{n-1}, g y_{n}\right)+d\left(y_{n}, g y_{n-1}\right)}\right\} \\
&\quad=\max \left\{d\left(y_{n-1}, y_{n}\right), \frac{d\left(y_{n-1}, y_{n}\right) d\left(y_{n}, y_{n+1}\right)}{1+d\left(y_{n}, y_{n+1}\right)}, \frac{d\left(y_{n-1}, y_{n}\right) d\left(y_{n}, y_{n+1}\right)}{1+d\left(y_{n-1}, y_{n}\right)},\right. \\
&\qquad \left.\frac{d\left(y_{n-1}, y_{n}\right) d\left(y_{n-1}, y_{n+1}\right)}{1+d\left(y_{n-1}, y_{n+1}\right)+d\left(y_{n}, y_{n}\right)}\right\} \\
&\quad\leq \max \left\{d\left(y_{n-1}, y_{n}\right), d\left(y_{n}, y_{n+1}\right)\right\}.
\end{aligned}
$$

If $\max \left\{d\left(y_{n-1}, y_{n}\right), d\left(y_{n}, y_{n+1}\right)\right\}=d\left(y_{n}, y_{n+1}\right)$, then we have

$$
d\left(y_{n}, y_{n+1}\right) \leq k d\left(y_{n}, y_{n+1}\right)
$$

for $k \in[0,1)$, which is a contradiction. Hence $\max \left\{d\left(y_{n-1}, y_{n}\right), d\left(y_{n}, y_{n+1}\right)\right\}= d\left(y_{n-1}, y_{n}\right)$. Therefore

$$
d\left(y_{n}, y_{n+1}\right) \leq k d\left(y_{n-1}, y_{n}\right) \leq k^{2} d\left(y_{n-2}, y_{n-1}\right) \leq \ldots \leq k^{n} d\left(y_{0}, y_{1}\right).
$$

It implies that

$$
\lim _{n \rightarrow \infty} d\left(y_{n}, y_{n+1}\right)=0. \tag{1.4}
$$

Step 2. Now, we show that the sequence $\{y_n\}$ g.m.s Cauchy sequence. Using the triangular inequality and by (1.1), we have

$$
\begin{aligned}
d\left(y_{n}, y_{m}\right) & \leq d\left(y_{n}, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)+d\left(y_{n+2}, y_{m}\right) \\
& \leq d\left(y_{n}, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)+d\left(y_{n+2}, y_{n+3}\right) \\
& \quad+d\left(y_{n+3}, y_{n+4}\right)+d\left(y_{n+4}, y_{m}\right) \\
& \leq k^{n} d\left(y_{0}, y_{1}\right)+k^{n+1} d\left(y_{0}, y_{1}\right)+\ldots+k^{m-1} d\left(y_{0}, y_{1}\right) \\
& =d\left(y_{0}, y_{1}\right)\left[k^{n}+k^{n+1}+\ldots+k^{m-1}\right] \\
& =d\left(y_{0}, y_{1}\right)\left[\frac{k^{n+1}}{k-1}-\frac{k^{m}-1}{k-1}\right].
\end{aligned} \tag{1.5}
$$

As $n, m \rightarrow \infty$, we obtain $d\left(y_{n}, y_{m}\right) \rightarrow 0$. Consequently, $\{y_n\}$ is a Cauchy sequence in $Y$. Since $(Y, d)$ is complete, the sequence $\{y_n\}$ converges to some $z \in Y$, that $\lim_{n \rightarrow \infty} d\left(y_{n}, z\right)=0$.

STEP 3. Now, we show that $z$ is a fixed point of $g$. Suppose that on the contrary $gz \neq z$. Then we can apply the rectangular inequality to obtain

$$\begin{aligned} d(gz, z) &\le d(gz, gy_n) + d(gy_n, gy_{n+1}) + d(gy_{n+1}, z) \\ &\le d(gz, gy_n) + d(gy_n, gy_{n+1}) + d(gy_{n+2}, z). \end{aligned}$$

As $n \to \infty$ and using the continuing of $g$ we have $gz = z$. Finally, suppose that the set of fixed point of $g$ is well ordered. Assume, on the contrary, that $u$ and $v$ are two fixed points of $g$, such that $u \neq v$. Then by (1.1) we have $d(u, v) = d(gu, gv) \le kM(u, v)$, where

$$\begin{aligned} M(u, v) = \max \Big\{ & d(u, v), \frac{d(u, gu)\, d(v, gv)}{1 + d(gu, gv)}, \frac{d(u, gu)\, d(v, gv)}{1 + d(u, v)}, \\ & \frac{d(u, gu)\, d(u, gv)}{1 + d(u, gv) + d(v, gu)} \Big\} \\ = \max & \{ d(u, v), 0 \} = d(u, v). \end{aligned}$$

So, we obtain $d(u, v) < kd(u, v)$, that is a contraction. Hence $u = v$ and $g$ has a unique fixed point. Conversely, if $g$ has a unique fixed point, then the set of fixed point of $g$ is a singleton and hence, it is well ordered. □

Notice that the continuing of $g$ in Theorem 1.4 can be replaced by another property.

**Theorem 1.5.** *Under the hypothesis of Theorem 1.4, without the continuing assumption on $g$, assume that whenever $\{y_n\}$ is a non-decreasing sequence in $Y$ such that $y_n \to y_0$ one has $y_n \preceq y_0$ for all $n$. Then $g$ has a fixed point.*

*Proof.* Repeating the proof of Theorem 1.4, we construct an increasing sequence $\{y_n\}$, with $y_n \neq y_m$ for all $m \neq n$ in $Y$ such that $y_n \to z \in Y$. Using the assumption on $Y$, we have $y_n \preceq z$. Now we show that $gz = z$. Suppose, on the contrary, that $gz \neq z$. By (1.1), we have

$$\begin{aligned} M(y_n, z) = \max \Big\{ & d(y_n, z), \frac{d(y_n, gy_n)\, d(z, gz)}{1 + d(gy_n, gz)}, \frac{d(y_n, gy_n)\, d(gz, gz)}{1 + d(y_n, z)}, \\ & \frac{d(y_n, gy_n)\, d(y_n, gz)}{1 + d(y_n, gz) + d(z, gy_n)} \Big\}. \end{aligned}$$

As $n \to \infty$, we have $\lim_{n\to\infty} M(y_n, z) = 0$. Therefore, we deduce that $d(z, gz) \le 0$, a contradiction. Hence we have $z = gz$. □

**Example 1.6.** Let $Y = \{a_1, a_2, a_3, a_4, a_5\}$ be equipped with the order $\preceq$ given by

$$\preceq = \big\{(a_1,a_1),(a_2,a_2),(a_3,a_3),(a_4,a_4),(a_5,a_5),(a_1,a_4),(a_1,a_5),(a_3,a_2),\\ (a_3,a_5),(a_1,a_2),(a_1,a_4),(a_4,a_5),(a_4,a_3),(a_5,a_2),(a_4,a_2)\big\},$$

and let $d : Y \times Y \to [0,\infty)$ be given as $d(x,x) = 0$ for $x \in Y$. $d(x,y) = d(y,x)$ for $x, y \in X$, $d(a_1,a_3) = d(a_1,a_5) = d(a_3,a_2) = d(a_3,a_5) = \frac{3}{4}$, $d(a_1,a_2) = 2$, $d(a_2,a_4) = \frac{8}{25}$, and $d(a_1,a_4) = d(a_4,a_5) = d(a_2,a_4) = d(a_2,a_5) = a(a_3,a_4) = \frac{2}{3}$. Then it is easy to check that $(Y,\preceq,d)$ is a complete ordered g.m.s. Consider the mapping $g : Y \to Y$ defined as

$$g = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_2 & a_2 & a_2 & a_4 & a_2 \end{pmatrix}.$$

If $x = a_1$ and $y \in \{a_2, a_3, a_4, a_5\}$, then we have $d(g(x,y) = d(a_2,a_4)$.
Case 1. If $x = a_1$ and $y = a_4$, then we have $M(a_1,a_4) = \max\left\{\frac{2}{3}, \frac{100}{141}\right\} = \frac{100}{141}$. So, we get $\frac{2}{3} \le k \times \frac{100}{141}$, hence $\frac{141}{150} \le k < 1$.
Case 2. If $x = a_1$ and $y = a_2$, then we have $d(gx,gy) = d(ga_1,ga_2) = d(a_2,a_2) = 0$.
Case 3. If $x = a_1$ and $y = a_3$, then we have $d(ga_1,ga_3) = d(a_2,a_2) = 0$.
Case 2. If $x = a_1$ and $y = a_5$, then we have $d(ga_1,ga_5) = d(a_2,a_2) = 0$.
It follows that $g$ has a unique fixed point which is $z = a_2$.

**Corollary 1.7.** *Let $(Y,\preceq,d)$ be a complete ordered g.m.s. Let $g : Y \to Y$ be an increasing mapping with respect to $\preceq$ such that there exists an element $x_0 \in Y$ with $x_0 \preceq gx_0$. Suppose that*

$$d(gx,gy) \le \mu_1 d(x,y) + \mu_2 \frac{d(x,gx)\,d(y,gy)}{1+d(gx,gy)} + \mu_3 \frac{d(x,gx)\,d(y,gy)}{1+d(x,y)} \\ + \mu_4 \frac{d(x,gx)\,d(x,gy)}{1+d(x,gy)+d(y,gx)},$$

*for all comparable elements $x, y \in Y$, where $\mu_1, \mu_2, \mu_3$, and $\mu_4 \ge 0$, with $\mu_1 + \mu_2 + \mu_3 + \mu_4 \le 1$. If $g$ is continuous, or for any non-decreasing sequence $\{x_n\}$ in $Y$ such that $x_n \to z \in Y$ one has $x_n \preceq z$ for all $n$, then $g$ has a fixed point.*

*Proof.* All the conditions of Theorem 1.4 hold, and hence, $g$ has a fixed point. □

**Corollary 1.8.** *Let $(Y,\preceq,d)$ be a complete totally ordered g.m.s and let $g : Y \to Y$ be an increasing mapping with respect to $\preceq$ such that there exists an element $y_0 \in Y$*

*with* $y_0 \preceq gy_0$*, Suppose that* $d(g^m x, g^m y) \leq kM(x,y)$*, where*

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,g^m x)\, d(y,g^m y)}{1+d(g^m x, g^m y)}, \frac{d(x,g^m x)\, d(y,g^m y)}{1+d(x,y)}, \frac{d(x,g^m x)\, d(x,g^m y)}{1+d(x,g^m y)+d(y,g^m x)}\right\}.$$

*If* $g^m$ *is continuous, or any non-decreasing sequence* $\{y_n\}$ *in* $Y$ *such that* $y_n \to z \in Y$*,* $y_n \preceq z$ *for all* $n$*, then* $g$ *has a fixed point.*

*Proof.* Since $g$ is an increasing mapping with respect to $\preceq$, $g^n$ also increasing mapping with respect to $\preceq$, and we have $y_0 \preceq gy_0 \preceq g^2y_0 \preceq \ldots \preceq g^m y_0$. Thus all condition of Theorem 1.4 hold for $g^m$ and it has a fixed point $z \in Y$, i.e, $g^m z = z$. Now we show that $gz = z$. If on the contrary, $gz \neq z$, then since the order $\preceq$ is total, we have $z < gz$ or $gz < z$. If $z < gz$, then we have $z \preceq gz \preceq g^2 z \preceq \ldots \preceq g^m z$, a contradiction. Similarly, for the case $gz < z$, we can get a contradiction. So $g$ has a fixed point. □

**1.2. Results in comparison functions** Let $\Psi$ denote the family of all non-decreasing and continuous functions $\psi : [0,\infty) \to [0,\infty)$ such that $\lim_{n\to\infty} \psi^n(t) = 0$ for all $t > 0$, where $\psi^n$ denotes the $n$th iterate of $\psi$. It is easy to show that for each $\psi \in \Psi$, the following is satisfied.

1) $\psi(t) < t$ for all $t > 0$;
2) $\psi(0) = 0$.

**Theorem 1.9.** *Let* $(Y, \preceq, d)$ *be a complete ordered g.m.s. Let* $g : Y \to Y$ *be an increasing mapping with respect to* $\preceq$ *such that there exists an element* $y_0 \in Y$ *with* $y_0 \preceq gy_0$*. Suppose that*

$$d(gx, gy) \leq \psi(M(x,y)), \tag{1.6}$$

*where*

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,gx)d(y,gy)}{1+d(gx,gy)}, \frac{d(x,gx)d(y,gy)}{1+d(x,y)}, \frac{d(x,gx)d(x,gy)}{1+d(x,gy)+d(y,gx)}\right\},$$

*for some* $\psi \in \Psi$ *and for all elements* $x, y \in Y$ *with* $x$ *and* $y$ *comparable. If* $g$ *is continuous, then* $g$ *has a fixed point. In addition, the set of fixed points of* $g$ *is well ordered if and only if* $g$ *has one and only one fixed point.*

*Proof.* Since $y_0 \preceq gy_0$ and $g$ is an increasing function, we obtain by induction that

$$y_0 \preceq gy_0 \preceq \cdots \preceq g^n y_0 \preceq g^{n+1} y_0 \leq \cdots .$$

By letting $y_n = g^n y_0$, we have

$$y_0 \preceq y_1 \preceq \cdots \preceq y_n \preceq y_{n+1} \preceq \cdots .$$

If there exists $n_0 \in \mathbb{N}$ such that $y_{n_0} = y_{n_0+1}$, then $y_{n_0} = gy_{n_0}$ and so, we have nothing prove. Hence, we assume $y_n \neq y_{n+1}$ for all $n$.

Step I. We will prove that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. Using condition (1.6), we obtain

$$d(y_n, y_{n+1}) = d(gy_{n-1}, gy_n) \leq \psi(M(y_{n-1}, y_n)),$$

where

$$\begin{aligned} M(y_{n-1}, y_n) &= \max\left\{ d(y_{n-1}, y_n), \frac{d(gy_{n-1}, gy_{n-1})\, d(y_n, gy_n)}{1 + d(gy_{n-1}, gy_n)} \right. \\ &\quad \left. \frac{d(y_{n-1}, gy_{n-1})\, d(y_n, gy_n)}{1 + d(y_{n-1}, y_n)}, \frac{d(y_{n-1}, gy_{n-1})\, d(y_{n-1}, gy_n)}{1 + d(y_{n-1}, gy_n) + d(y_n, gy_{n-1})} \right\} \\ &= \max\left\{ d(y_{n-1}, y_n), \frac{d(y_{n-1}, y_n)\, d(y_n, y_{n+1})}{1 + d(y_n, y_{n+1})} \right. \\ &\quad \left. \frac{d(y_{n-1}, y_n)\, d(y_n, y_{n+1})}{1 + d(y_{n-1}, y_n)}, \frac{d(y_{n-1}, y_n)\, d(y_{n-1}, y_{n+1})}{1 + d(y_{n-1}, y_{n+1}) + 0} \right\} \\ &\leq \max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\}. \end{aligned}$$

If $\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_n, y_{n+1})$, then we have

$$d(y_n, y_{n+1}) \leq \psi(d(y_n, y_{n+1})) < d(y_n, y_{n+1}).$$

It is a contradiction. Thus $\max\{d(y_{n-1}, y_n), d(y_n, y_{n+1})\} = d(y_{n-1}, y_n)$ and we obtain $d(y_n, y_{n+1}) \leq \psi(d(y_{n-1}, y_n))$. By induction, we have

$$d(y_n, y_{n+1}) < \psi(d(y_{n-1}, y_n)) < d(y_{n-1}, y_n) \leq \psi(d(y_{n-2}, y_{n-1})),$$

and so

$$d(y_n, y_{n+1}) < \psi^2(d(y_{n-2}, y_{n-1})) \leq \psi^3(d(y_{n-3}, y_{n-2})) \leq \ldots \leq \psi^3(d(y_0, y_1)).$$

$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Step II. Suppose that $y_n = y_m$ for some $m, n \in \mathbb{N}$ with $m < n$, then

$$\begin{aligned} d(y_m, y_{m+1}) &= d(y_n, y_{n+1}) \\ &\leq \psi^{n-m}(d(y_m, y_{m+1})) \\ &< d(y_m, y_{m+1}), \end{aligned}$$

a contradiction. Hence all elements of the Picard sequence $\{y_n\}$ are distinct. Now, we prove that $\{y_n\}$ is a g.m.s Cauchy sequence. We have

$$\begin{aligned}
d(y_n, y_m) &\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_m) \\
&\leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + d(y_{n+2}, y_{n+3}) + \ldots + d(y_{m-1}, y_m) \\
&\leq \psi^n(d(y_0, y_1)) + \psi^{n+1}(d(y_0, y_1)) + \ldots + \psi^{m-1}(d(y_0, y_1)).
\end{aligned}$$

Suppose $\psi^n(d(y_0, y_1)) = t$, then we have

$$d(y_n, y_m) \leq t + \psi(t) + \psi^2(t) + \ldots + \psi^{m-n-1}(t).$$

Since $\lim_{n\to\infty} \psi^n(d(y_0, y_1)) = 0$,

$$\lim_{n,m\to\infty} \psi(t) = \psi^2(t) = \ldots = \psi^{m-n-1}(t) = 0.$$

Therefore the sequence $\{y_n\}$ Cauchy g.m.s sequence and the sequence $\{y_n\}$ converges to some $z \in Y$ is, $\lim_{n\to\infty} d(y_n, z) = 0$.

STEP III. We show that $z$ is a fixed point of $g$. Suppose, on the contrary, that $gz \neq z$, then by Lemma 1.3 it follows that $y_n$ differs from both $gz$ and $z$ for $n$ sufficiently large. Using the rectangular inequality, we get

$$d(z, gz) \leq d(z, gy_n) + d(gy_n, gy_{n+1}) + d(gy_{n+1}, gy_z).$$

Letting $n \to \infty$ and using the continuity of $g$, we get $d(z, gz) \leq 0$. Hence we have $gz = z$. Thus, $z$ is a fixed point of $g$. □

**Theorem 1.10.** *Under the hypothesis of Theorem 1.9 without the continuity assumption of $g$, assume that whenever $\{y_n\}$ is a non-decreasing sequence in $Y$ such that $x_n \to z \in Y$ one has $y_n \preceq u$ for all $n$. Then $g$ has a fixed point.*

*Proof.* Following the proof of Theorem 1.9, we construct an increasing sequence $\{y_n\}$ in $Y$, such that $y_n \to z \in Y$. Using the given assumption on $Y$, we have $y_n \preceq z$. Now we show that $z = gz$. By (1.6) we have

$$d(gz, y_n) = d(gz, gy_{n-1}) \leq \psi(M(z, y_{n-1})), \tag{1.7}$$

where

$$M(y_{n-1}, z) = \max\left\{d(y_{n-1}, z), \frac{d(y_{n-1}, gy_{n-1})\, d(z, gz)}{1 + d(gy_{n-1}, gz)}, \frac{d(y_{n-1}, gy_{n-1})\, d(z, gz)}{1 + d(y_{n-1}, z)}, \frac{d(z, gz)\, d(z, gy_{n-1})}{1 + d(y_{n-1}, gz) + d(z, gy_{n-1})}\right\}.$$

As $n \to \infty$, we get

$$\limsup_{n\to\infty} M\left(y_{n-1}, z\right) = 0. \tag{1.8}$$

Again, taking the upper limit as $n \to \infty$ in (1.7) and using (1.8), we obtain

$$d\left(gz, z\right) \leq \limsup_{n\to\infty} d\left(gz, y_n\right) \leq \limsup_{n\to\infty} \psi\left(M(z, y_{n-1})\right) = 0,$$

or $d\left(gz, z\right) \leq 0$. So, we get $d\left(gz, z\right) = 0$, i.e, $g(z) = z$. □

That Khan et, introduced the concept of an altering distance function as follows.

**Definition 1.11** ([7])**.** A function $\varphi : [0, \infty) \to [0, \infty)$ is called an *altering distance function*, if the following properties hold

1) $\varphi$ is continuous and non-decreasing;
2) $\varphi(t) = 0$ if and only if $t = 0$.

**Definition 1.12.** Let $(Y, d)$ be a g.m.s, and let $g : Y \to Y$ be a mapping. For $x, y \in Y$, set

$$M(x, y) = \max\left\{d(x, y), d(x, gx), d(y, gy)\right\},$$

and

$$N(x, y) = \min\left\{d(x, gx), d(y, gy), d(x, gy), d(y, gx)\right\}.$$

We say that $g$ is an *almost generalized* $(\psi, \varphi)$*-contractive mapping* if there $L \geq 0$, $k \in [0, 1)$, and two altering distance functions $\psi$ and $\varphi$ such that

$$\psi\left(d\left(gx, gy\right)\right) \leq \psi\left(kM(x, y)\right) - \varphi\left(kM(x, y)\right) + L\psi\left(kN(x, y)\right), \tag{1.9}$$

for all $x, y \in Y$.

**Theorem 1.13.** *Let $(Y, \preceq, d)$ be a complete ordered g.m.s and Let $g : Y \to Y$ be a continuous mapping which is non-decreasing with respect to $\preceq$. Suppose that $g$ satisfies condition* (1.9) *for all elements $x, y \in Y$ with $x, y$ comparable. If there exists $y_0 \in Y$ such that $y_0 \preceq gy_0$, then $g$ has a fixed point. Moreover, the set of fixed points of $g$ is well ordered if and only if $g$ has one and only one fixed point.*

*Proof.* Starting with given $y_0$, define a sequence $\{y_n\}$ in $Y$ such that $y_{n+1} = gy_n$ for all $n \geq 0$. Since $y_0 \preceq gy_0 = y_1$ and $g$ is non-decreasing, we have $y_1 = gy_0 \preceq y_2 = gy_1$, and by induction

$$y_0 \preceq y_1 \preceq \cdots \preceq y_n \preceq y_{n+1} \preceq \cdots .$$

We will again assume that $y_n \neq y_{n+1}$ for each $n$, By (1.9) we have

$$\begin{aligned} \psi\left(d\left(y_n, y_{n+1}\right)\right) &= \psi\left(d\left(g y_{n-1}, g y_n\right)\right) \\ &\leq \psi\left(k M\left(y_{n-1}, y_n\right)\right)-\varphi\left(k M\left(y_{n-1}, y_n\right)\right) \\ &\quad + L \psi\left(k N\left(y_{n-1}, y_n\right)\right), \end{aligned} \tag{1.10}$$

where

$$\begin{aligned} M\left(y_{n-1}, y_n\right) &= \max \left\{d\left(y_{n-1}, y_n\right), d\left(y_{n-1}, g y_{n-1}\right), d\left(y_n, g y_n\right)\right\} \\ &= \max \left\{d\left(y_{n-1}, y_n\right), d\left(y_n, y_{n+1}\right)\right\}, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} N\left(y_{n-1}, y_n\right) &= \min \left\{d\left(y_{n-1}, g y_{n-1}\right), d\left(y_{n-1}, g y_n\right), d\left(y_n, g y_{n-1}\right), d\left(y_n, g y_n\right)\right\} \\ &= \min \left\{d\left(y_{n-1}, y_n\right), d\left(y_{n-1}, y_{n+1}\right), d\left(y_n, y_n\right), d\left(y_n, y_{n+1}\right)\right\}=0. \end{aligned} \tag{1.12}$$

From (1.10), (1.11) and (1.12) and the properties of $\psi$ and $\varphi$, we get

$$\psi\left(d\left(y_n, y_{n+1}\right)\right)<\psi\left(k \max \left\{d\left(y_{n-1}, y_n\right), d\left(y_n, y_{n+1}\right)\right\}\right).$$

We conclude

$$d\left(y_n, y_{n+1}\right)<k \max \left\{d\left(y_{n-1}, y_n\right), d\left(y_n, y_{n+1}\right)\right\}. \tag{1.13}$$

If $\max \left\{d\left(y_{n-1}, y_n\right), d\left(y_n, y_{n+1}\right)\right\}=d\left(y_n, y_{n+1}\right)$, then we get $d\left(y_n, y_{n+1}\right)<k d\left(y_n, y_{n+1}\right)$, a contradiction. Hence $\max \left\{d\left(y_{n-1}, y_n\right), d\left(y_n, y_{n+1}\right)\right\}=d\left(y_{n-1}, y_n\right)$. Therefore, $d\left(y_n, y_{n+1}\right)<k d\left(y_{n-1}, y_n\right)$. By induction, we have

$$d\left(y_n, y_{n+1}\right)<k d\left(y_{n-1}, y_n\right)<k^2 d\left(y_{n-2}, y_{n-1}\right)<\ldots<k^n d\left(y_0, y_1\right),$$

implies that $\lim_{n \rightarrow \infty} d\left(y_n, y_{n+1}\right)=0$. We show that $\{y_n\}$ is a g.m.s Cauchy sequence in $Y$. By rectangular inequality for $n<m$, we have

$$d\left(y_n, y_m\right) \leq d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)+d\left(y_{n+2}, y_m\right),$$

and

$$\begin{aligned} d\left(y_n, y_m\right) &\leq d\left(y_n, y_{n+1}\right)+d\left(y_{n+1}, y_{n+2}\right)+\ldots+d\left(y_{m-1}, y_m\right) \\ &\leq k^n d\left(y_0, y_1\right)+k^{n+1} d\left(y_0, y_1\right)+\ldots+k^{m-1} d\left(y_0, y_1\right) \\ &= d\left(y_0, y_1\right)\left[k^n+k^{n+1}+\ldots+k^{m-1}\right] \\ &= k^n d\left(y_0, y_1\right)\left[1+k+\ldots+k^{m-n-1}\right]. \end{aligned}$$

As $n, m \rightarrow \infty$, we have $\lim_{n, m \rightarrow \infty} d\left(y_n, y_m\right)=0$ and sequence $\{y_n\}$ g.m.s Cauchy sequence. As $Y$ is complete, then there exists $z \in Y$ such that $y_n \rightarrow z$ as $n \rightarrow \infty$, that is $\lim_{n \rightarrow \infty} y_{n+1}=\lim_{n \rightarrow \infty} g y_n=z$. Now, suppose that $g$ is continuous, we show that $z$ is a fixed point of $g$. Suppose, on the contrary, that $gz \neq z$. Then by Lemma 1.3, it follows that $y_n$ differs from both $gz$ and $z$ for $n$ sufficiently large. Using the rectangular inequality we obtain

$$d(z, gz) \leq d(z, gy_n) + d(gz, gy_{n+1}) + d(gy_{n+1}, gz).$$

As $n \to \infty$, we get $d(z, gz) \leq 0$. So, we have $gz = z$. Thus, $z$ is a fixed point of $g$. □

The continuity of $g$ can be replaced by another condition.

**Theorem 1.14.** *Under the hypothesis of Theorem 1.13 without the continuity on $g$, assume that whenever $\{y_n\}$ is a non-decreasing sequence in $Y$ such that $y_n \to z \in Y$ one has $y_n \preceq z$, for all $n$. Then $g$ has a fixed point in $Y$.*

*Proof.* Following similar arguments to those given in the proof of Theorem 1.13, we construct an increasing sequence $\{y_n\}$ in $Y$ such that $y_n \to z$ for some $z \in Y$. Using the assumption on $Y$, we have that $y_n \preceq z$ for all $n$. Now, we show that $gz = z$. By (1.9), we have

$$\begin{aligned}\psi(d(y_{n+1}, gz)) &= \psi(d(gy_n, gz)) \\ &\leq \psi(kM(y_n, z)) - \varphi(kM(y_n, z)) + L\psi(kN(y_n, z)),\end{aligned}$$

where

$$M(y_n, z) = \max\left\{d(y_n, z), d(y_n, gy_n), d(z, gz)\right\},$$

and

$$N(y_n, z) = \min\left\{d(y_n, y_{n+1}), d(y_n, gz), d(z, y_{n+1}), d(z, gz)\right\}.$$

Letting $n \to \infty$, we get $M(y_n, z) \to d(z, gz)$ and $N(y_n, z) \to 0$.

$$\begin{aligned}\psi(d(z, gz)) &\leq \psi(kd(z, gz)) - \varphi\left(k \liminf_{n\to\infty} M(y_n, z)\right) \\ &\leq \psi(d(z, gz)) - \varphi\left(k \liminf_{n\to\infty} M(y_n, z)\right).\end{aligned}$$

Therefore, $\varphi(k \liminf_{n\to\infty} M(y_n, z)) \leq 0$, equivalently $k \liminf_{n\to\infty} M(y_n, z) = 0$. Thus by $M(y_n, z) \to d(z, gz)$, we deduce that $d(z, gz) = 0$ and hence $z$ is a fixed point of $g$. □

**Corollary 1.15.** *Let $(Y, \preceq, d)$ be a complete ordered g.m.s and let $g : Y \to Y$ be a non-decreasing continuous mapping with respect to $\preceq$. Suppose that there $k \in [0, 1)$*

*and $L \geq 0$ such that*

$$d(gx, gy) \leq k \max\left\{d(x, y), d(x, gx), d(y, gy)\right\} + L \min\left\{d(x, gx), d(x, gx), d(y, gx), d(y, gy)\right\},$$

*for all elements $x, y \in Y$ with $x, y$ comparable. If there exists $x_0 \in Y$ such that $y_0 \preceq gy_0$, then $g$ has a fixed point provided that*

(i) *$g$ is continuous, or*

(ii) *for any non-decreasing sequence $\{y_n\}$ in $Y$ such that $y_n \to z \in Y$, we have $y_n \preceq z$ for all $n \in \mathbb{N}$.*

**Example 1.16.** Consider $Y = B \cup [1, 2]$, where $B = \left\{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ endowed with the order defined as follows

$$t \preceq \frac{1}{3} \preceq \frac{1}{5} \preceq \frac{1}{2} \preceq 0 \preceq \frac{1}{4},$$

for all $t \in [1, 2]$. Define $d : Y \times Y \to [0, \infty)$ as follows $d(x, x) = 0$ for all $x \in Y$, $d(x, y) = d(y, x)$ for all $x, y, \in X$, and

$$d(0, \frac{1}{3}) = d(0, \frac{1}{5}) = d(\frac{1}{3}, \frac{1}{2}) = d(\frac{1}{3}, \frac{1}{5}) = 0.75,$$
$$d(0, \frac{1}{2}) = 2, \quad d(\frac{1}{2}, \frac{1}{4}) = 0.32,$$
$$d(0, \frac{1}{4}) = d(\frac{1}{4}, \frac{1}{5}) = d(\frac{1}{4}, \frac{1}{2}) = d(\frac{1}{2}, \frac{1}{3}) = d(\frac{1}{4}, \frac{1}{3}) = d(\frac{1}{5}, \frac{1}{2}) = 0.66.$$

If $\{x, y\} \cap Y = \emptyset$, $d(x, y) = |x - y|$. Clearly $(Y, d)$ is a generalized metric space. Consider now the mapping $g : Y \to Y$ given as

$$g(x) = \begin{cases} \frac{1}{5}, & x \in [1, 2], \\ \frac{1}{2}, & x \in A \setminus \{\frac{1}{4}\}, \\ \frac{1}{4}, & x = \frac{1}{4}. \end{cases}$$

It is easy to check that $g$ is increasing respect to $\preceq$ and there exists $x_0 \in Y$ such that $x_0 \preceq gx_0$ and the contraction condition in Corollary 1.15 with $k = 0.49$ is fulfilled, and $g$ hs a unique fixed point which is $z = \frac{1}{2}$.

## Conclusion

Diverse toppics had appeared as the results of the using comparision functions and almost weakly contractive mapping for new fixed points in generalized metric space.

## Abbreviations

Nfprc, new fixed point results comparison. Wcf, weakly contractinate function.

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PROFESSOR: DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, BU-ALI SINA UNIVERSITY, HAMEDAN 65178, IRAN
*Email address*: gh_khalilzadeh@yahoo.com