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COMPARATIVE GROWTH PROPERTIES OF ENTIRE AND MEROMORPHIC FUNCTIONS CONCERNING RELATIVE (α, β, γ)-ORDER

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Abstract. Belaïdi et al. [3] have introduced the idea (α, β, γ)-order and (α, β, γ)-lower order of meromorphic function, where $\alpha \in L_1$-class, $\beta \in L_2$-class, $\gamma \in L_3$-class. In order to make some progresses in the study of growth analysis of meromorphic functions, here in this paper, we have discussed on the relative (α, β, γ)-order and relative (α, β, γ)-lower order of a meromorphic function with respect to an entire function. Then we have investigated some basic properties of meromorphic functions using these definitions under somewhat different conditions.

1. Introduction

The standard notations of the Nevanlinna value distribution theory of entire and meromorphic functions are available in [5, 7, 8, 9, 10], so we do not explain those in details. For $x \in [0, \infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ be the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$. For meromorphic function f, the Nevanlinna's characteristic function $T_f(r)$ is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

where $m_f(r)$ and $N_f(r)$ are respectively called as the proximity function of f and the counting function of poles of f in $|z| \leq r$. For details about $T_f(r)$, $m_f(r)$ and $N_f(r)$, one may see [5, p. 4]. If f is an entire function, then the Nevanlinna's characteristic

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function $T_f(r)$ is defined as

$$T_f(r) = m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})|d\theta, \text{ where}$$
$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0.$$

Moreover, if f is non-constant entire function, then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0), \infty) \to (0, \infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. To start our paper, we just recall the following definition:

Definition 1.1. The order ρ_f and the lower order λ_f of a meromorphic function f are defined as:

$$\rho_f = \limsup_{r\to+\infty} \frac{\log T(r, f)}{\log r} \text{ and } \lambda_f = \liminf_{r\to+\infty} \frac{\log T(r, f)}{\log r}.$$

Now first of all, let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0, 1]$,

$$\begin{aligned} \alpha(tx) &= \alpha(tx + (1 - t) \cdot 0) \\ &\geq t\alpha(x) + (1 - t)\alpha(0) \geq t\alpha(x), \end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$, we obtain

$$\begin{aligned} \alpha(a + b) &= \frac{a}{a + b}\alpha(a + b) + \frac{b}{a + b}\alpha(a + b) \\ &\leq \alpha\left(\frac{a}{a + b}(a + b)\right) + \alpha\left(\frac{b}{a + b}(a + b)\right) \\ &= \alpha(a) + \alpha(b), \ a, b \geq 0. \end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r+R_0)$ as $r \to +\infty$. Throughout this paper, we assume $\alpha \in L_1$, $\beta \in L_2$, $\gamma \in L_3$.

Heittokangas et al. [6] have introduced the concept of φ-order of entire and meromorphic functions considering φ as subadditive function. For details one may see [6]. Later on Belaïdi et al. [3] have extended the above idea and have introduced the definitions of (α,β,γ)-order and (α,β,γ)-lower order of a meromorphic function f, which are as follows:

Definition 1.2 ([3])**.** The (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ and (α,β,γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$, of a meromorphic function f, are defined as:

$$\begin{aligned} \rho_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} \\ \text{and } \lambda_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\log(T_f(r)))}{\beta\left(\log(\gamma(r))\right)}. \end{aligned}$$

Mainly, the growth investigation of meromorphic function has usually been done through the Nevanlinna's characteristic function comparing with the exponential function. But if one is paying attention to evaluate the growth rates of any meromorphic function with respect to a entire function, the notions of relative growth indicators (see e.g. [1, 2]) will come. Now in order to make some progresses in the study of relative order of meromorphic function, Biswas et al. [4] have introduced the definitions of relative (α,β,γ)-order and relative (α,β,γ)-lower order of a meromorphic function with respect to an entire function in the following way:

Definition 1.3 ([4])**.** The relative (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]_h$ and relative (α,β,γ)-lower order denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ of a meromorphic function f with respect to an entire function h are defined as:

$$\begin{aligned} \rho_{(\alpha,\beta,\gamma)}[f]_h &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)} \\ \text{and } \lambda_{(\alpha,\beta,\gamma)}[f]_h &= \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} T_h^{-1}(T_f(r)))}{\beta\left(\log(\gamma(r))\right)}. \end{aligned}$$

Remark 1.4. Clearly if $h(z) = z$, then Definition 1.3 reduces to Definition 1.2. If we take $\alpha(r) = \beta(r) = \gamma(r) = r$ and $h(z) = z$, then Definition 1.3 reduces to Definition 1.1.

Remark 1.5. An entire function f is said to have *regular relative* (α,β,γ)*-order* with respect to an entire function h if $\rho_{(\alpha,\beta,\gamma)}[f]_h = \lambda_{(\alpha,\beta,\gamma)}[f]_h$.

Here, in this paper, we aim at investigating some basic properties of relative (α,β,γ)-order and relative (α,β,γ)-lower order of meromorphic functions with respect to entire functions under somewhat different conditions. Throughout this paper, we assume that all the growth indicators are nonzero finite.

2. Main Results

In this section, we present the main results of the paper.

Theorem 2.1. *Let f and g be two meromorphic functions and h be a non-constant entire function such that at least f or g is of regular relative (α,β,γ)-order with respect to h, then*

$$\lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h \le \max\{\lambda_{(\alpha,\beta,\gamma)}[f]_h,\ \lambda_{(\alpha,\beta,\gamma)}[g]_h\}.$$

The equality holds when either (i) $\lambda_{(\alpha,\beta,\gamma)}[f]_h > \lambda_{(\alpha,\beta,\gamma)}[g]_h$ with g has regular relative (α,β,γ)-order with respect to h or (ii) $\lambda_{(\alpha,\beta,\gamma)}[g]_h > \lambda_{(\alpha,\beta,\gamma)}[f]_h$ with f has regular relative (α,β,γ)-order with respect to h.

Proof. If $\lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h = 0$, then the result is obvious. So we suppose that $\lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h > 0$. Clearly, both $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ and $\lambda_{(\alpha,\beta,\gamma)}[g]_h$ are finite. Let us assume that $\max\{\lambda_{(\alpha,\beta,\gamma)}[f]_h,\ \lambda_{(\alpha,\beta,\gamma)}[g]_h\} = \Delta$ and f has regular relative (α,β,γ)-order with respect to h, then for any arbitrary $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ $(= \rho_{(\alpha,\beta,\gamma)}[f]_h)$, we have for all sufficiently large values of r that

$$T_f(r) \le T_h\left(\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right). \tag{2.1}$$

Again for $\varepsilon > 0$ from the definition of $\lambda_{(\alpha,\beta,\gamma)}[g]_h$, we have for a sequence values of r tending to infinity that

$$T_g(r) \le T_h\left(\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[g]_h+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right). \tag{2.2}$$

Since $T_{f\pm g}(r) \le T_f(r)+T_g(r)+\log 2$ for all large values of r, so in view of (2.1) and (2.2), we obtain for a sequence values of r tending to infinity that

$$\begin{aligned} T_{f\pm g}(r) \ &\le\ T_h\left(\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f]_h+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right) \\ &\quad +T_h\left(\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[g]_h+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right)+\log 2, \end{aligned}$$

$$i.e.,\ T_{f\pm g}(r) \le 2T_h\left(\exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right)+\log 2,$$

$$i.e.,\ T_{f\pm g}(r) \le 3T_h\left(\exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right). \tag{2.3}$$

So, in view of (2.3), we obtain for a sequence values of r tending to infinity that

$$T_{f\pm g}(r) \le T_h\left(\exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right)^3,$$

$$i.e.,\ T_h^{-1}T_{f\pm g}(r) \le \left(\exp^{[2]}(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)])\right)^3,$$

$$\begin{aligned} i.e.,\ \log T_h^{-1}T_{f\pm g}(r) &\le 3\exp(\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\\ i.e.,\ \log^{[2]} T_h^{-1}T_{f\pm g}(r) &\le \log 3 + (\alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)]),\end{aligned}$$

$$\begin{aligned} i.e.,\ \log^{[2]} T_h^{-1}T_{f\pm g}(r) - \log 3 &\le \alpha^{-1}[(\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right)],\\ i.e.,\ (1-o(1))\alpha\left(\log^{[2]} T_h^{-1}T_{f\pm g}(r)\right) &\le (\Delta+\varepsilon)\cdot\beta\left(\log(\gamma(r))\right).\end{aligned}$$

Thus for a sequence values of r tending to infinity, we get that

$$\frac{(1-o(1))\alpha\left(\log^{[2]} T_h^{-1}T_{f\pm g}(r)\right)}{\beta\left(\log(\gamma(r))\right)} \le \Delta+\varepsilon.$$

Hence,

$$\liminf_{r\to+\infty}\frac{(1-o(1))\alpha\left(\log^{[2]} T_h^{-1}T_{f\pm g}(r)\right)}{\beta\left(\log(\gamma(r))\right)} \le \Delta+\varepsilon,$$

$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h \le \Delta+\varepsilon.$$

As $\varepsilon>0$ is arbitrary,

$$\lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h \le \Delta.$$

Hence,

$$\lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h \le \max\{\lambda_{(\alpha,\beta,\gamma)}[f]_h, \lambda_{(\alpha,\beta,\gamma)}[g]_h\}.$$

This completes the proof of first part of the theorem.

Next let $\lambda_{(\alpha,\beta,\gamma)}[f]_h > \lambda_{(\alpha,\beta,\gamma)}[g]_h$ with g has regular relative (α,β,γ)-order with respect to h. Taking $f\pm g=k$, then we get from first part,

$$\lambda_{(\alpha,\beta,\gamma)}[k]_h \le \lambda_{(\alpha,\beta,\gamma)}[f]_h. \tag{2.4}$$

Also, we have $f=k\mp g$. So again, by using first part,

$$\begin{aligned} \lambda_{(\alpha,\beta,\gamma)}[f]_h &\le \max\{\lambda_{(\alpha,\beta,\gamma)}[k]_h, \lambda_{(\alpha,\beta,\gamma)}[g]_h\},\\ i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f]_h &\le \lambda_{(\alpha,\beta,\gamma)}[k]_h. \qquad (2.5)\end{aligned}$$

From (2.4) and (2.5), we have $\lambda_{(\alpha,\beta,\gamma)}[k]_h = \lambda_{(\alpha,\beta,\gamma)}[f]_h$,

$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f\pm g]_h = \max\{\lambda_{(\alpha,\beta,\gamma)}[f]_h, \lambda_{(\alpha,\beta,\gamma)}[g]_h\}.$$

This completes the proof. □

In line of Theorem 2.1 one can easily prove the following theorem, so we omit its proof.

Theorem 2.2. *Let f and g be two meromorphic functions and h be a non-constant entire function such that $\rho_{(\alpha,\beta,\gamma)}[f]_h$ and $\rho_{(\alpha,\beta,\gamma)}[g]_h$ exist, then*

$$\rho_{(\alpha,\beta,\gamma)}[f \pm g]_h \leq \max\{\rho_{(\alpha,\beta,\gamma)}[f]_h,\ \rho_{(\alpha,\beta,\gamma)}[g]_h\}.$$

The equality holds when $\rho_{(\alpha,\beta,\gamma)}[f]_h \neq \rho_{(\alpha,\beta,\gamma)}[g]_h$.

Theorem 2.3. *Let f be a meromorphic function and g, h be two non-constant entire functions such that $\lambda_{(\alpha,\beta,\gamma)}[f]_g$ and $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ exist, then*

$$\lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h} \geq \min\{\lambda_{(\alpha,\beta,\gamma)}[f]_g,\ \lambda_{(\alpha,\beta,\gamma)}[f]_h\}..$$

The equality holds when $\lambda_{(\alpha,\beta,\gamma)}[f]_g \neq \lambda_{(\alpha,\beta,\gamma)}[f]_h$.

Proof. If $\lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h} = \infty$ then the result is obvious. So we suppose that $\lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h} < \infty$. Clearly, both $\lambda_{(\alpha,\beta,\gamma)}[f]_g$ and $\lambda_{(\alpha,\beta,\gamma)}[f]_h$ are finite. Let us assume that $\min\{\lambda_{(\alpha,\beta,\gamma)}[f]_g, \lambda_{(\alpha,\beta,\gamma)}[f]_h\} = \Delta$, then for any arbitrary $\varepsilon > 0$ from the definitions of $\lambda_{(\alpha,\beta,\gamma)}[f]_g$ and $\lambda_{(\alpha,\beta,\gamma)}[f]_h$, we have for all sufficiently large values of r that

$$\begin{aligned} T_f(r) &\geq T_g\left(\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f]_g - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])\right), \\ \text{i.e., } T_f(r) &\geq T_g\left(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])\right). \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} T_f(r) &\geq T_h\left(\exp^{[2]}(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f]_h - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])\right), \\ \text{i.e., } T_f(r) &\geq T_h\left(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])\right). \end{aligned} \tag{2.7}$$

Since $T_{f\pm g}(r) \leq T_f(r) + T_g(r) + \log 2$, so in view of (2.6) and (2.7), we obtain for all sufficiently large values of r that

$$\begin{aligned} & T_{g\pm h}(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])) \\ \leq\ & T_g\left(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])\right) \\ & + T_h\left(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])\right) + \log 2, \end{aligned}$$

$$\text{i.e., } T_{g\pm h}(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])) \leq 3T_f(r). \tag{2.8}$$

Therefore, from (2.8) we obtain for all sufficiently large values of r that

$$T_{g\pm h}(\exp^{[2]}(\alpha^{-1}[(\Delta - \varepsilon) \cdot \beta\left(\log(\gamma(r))\right)])) \leq T_f(r^3)$$

$$i.e.,\ \Delta-\varepsilon \leq \frac{\alpha\left(\log^{[2]} T^{-1}_{g\pm h}T_f(r^3)\right)}{\beta\left(\log(\gamma(r))\right)},$$

Hence,

$$\liminf_{r\to+\infty}\left[\frac{\alpha\left(\log^{[2]} T^{-1}_{g\pm h}T_f(r^3)\right)}{\beta\left(\log(\gamma(r^3))\right)}\cdot\frac{\beta\left(\log(\gamma(r^3))\right)}{\beta\left(\log(\gamma(r))\right)}\right] \geq \Delta-\varepsilon,$$

$$\liminf_{r\to+\infty}\frac{\alpha\left(\log^{[2]} T^{-1}_{g\pm h}T_f(r^3)\right)}{\beta\left(\log(\gamma(r^3))\right)}\cdot\lim_{r\to+\infty}\frac{\beta\left(\log(\gamma(r^3))\right)}{\beta\left(\log(\gamma(r))\right)} \geq \Delta-\varepsilon,$$

$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h}\geq\Delta-\varepsilon.$$

As $\varepsilon>0$ is arbitrary,

$$\lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h}\geq\Delta.$$

Hence,

$$\lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h}\geq\min\{\lambda_{(\alpha,\beta,\gamma)}[f]_g,\lambda_{(\alpha,\beta,\gamma)}[f]_h\}.$$

This completes the proof of first part of the theorem.

Next let $\lambda_{(\alpha,\beta,\gamma)}[f]_g\neq\lambda_{(\alpha,\beta,\gamma)}[f]_h$, and without loss of generality, we assume that $\lambda_{(\alpha,\beta,\gamma)}[f]_g>\lambda_{(\alpha,\beta,\gamma)}[f]_h$. Taking $g\pm h=k$, then we get from first part,

$$\lambda_{(\alpha,\beta,\gamma)}[f]_k\geq\lambda_{(\alpha,\beta,\gamma)}[f]_h. \tag{2.9}$$

Also, we have $h=k\mp g$. So,

$$\begin{aligned}\lambda_{(\alpha,\beta,\gamma)}[f]_h &\geq \min\{\lambda_{(\alpha,\beta,\gamma)}[f]_k,\lambda_{(\alpha,\beta,\gamma)}[f]_g\},\\ i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f]_h &\leq \lambda_{(\alpha,\beta,\gamma)}[f]_k. \qquad (2.10)\end{aligned}$$

From (2.9) and (2.10), we have $\lambda_{(\alpha,\beta,\gamma)}[f]_k=\lambda_{(\alpha,\beta,\gamma)}[f]_h$,

$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f]_{g\pm h}=\min\{\lambda_{(\alpha,\beta,\gamma)}[f]_g,\lambda_{(\alpha,\beta,\gamma)}[f]_h\}.$$

This completes the proof. □

In line of Theorem 2.3 one can easily prove the following theorem, so we omit its proof.

Theorem 2.4. *Let f be a meromorphic function and g, h be two non-constant entire functions such that f has regular relative (α,β,γ)-order with respect to g or h, then*

$$\rho_{(\alpha,\beta,\gamma)}[f]_{g\pm h}\geq\min\{\rho_{(\alpha,\beta,\gamma)}[f]_g,\ \rho_{(\alpha,\beta,\gamma)}[f]_h\}.$$

The equality holds when either (i) $\rho_{(\alpha,\beta,\gamma)}[f]_g>\rho_{(\alpha,\beta,\gamma)}[f]_h$ with f has regular relative (α,β,γ)-order with respect to g or (ii) $\rho_{(\alpha,\beta,\gamma)}[f]_h>\rho_{(\alpha,\beta,\gamma)}[f]_g$ with f has regular relative (α,β,γ)-order with respect to h.

Theorem 2.5. *Let f_1, f_2 be two meromorphic functions and g_1, g_2 be two entire functions such that the following conditions are satisfied:*

(i) $\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_i} < \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_j}$ with at least f_1 is of regular relative (α,β,γ)-order with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and

(ii) $\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_i} < \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_j}$ with at least f_2 is of regular relative (α,β,γ)-order with respect to g_j for $i = 1, 2$, $j = 1, 2$ and $i \neq j$. Then

$$\begin{aligned} & \varrho_{(\alpha,\beta,\gamma)}[f_1 \pm f_2]_{g_1 \pm g_2} \\ \leq \ & \max[\min\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\}]. \end{aligned}$$

The equality holds when $\varrho_{(\alpha,\beta,\gamma)}[f_i]_{g_1} < \varrho_{(\alpha,\beta,\gamma)}[f_j]_{g_1}$ and $\varrho_{(\alpha,\beta,\gamma)}[f_i]_{g_2} < \varrho_{(\alpha,\beta,\gamma)}[f_j]_{g_2}$ holds simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Proof. Suppose that the conditions (i) and (ii) of the theorem holds. Therefore in view of Theorem 2.2 and Theorem 2.4 we get that

$$\begin{aligned} & \max[\min\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\}] \\ & = \max[\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \pm g_2}, \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1 \pm g_2}] \\ & \geq \varrho_{(\alpha,\beta,\gamma)}[f_1 \pm f_2]_{g_1 \pm g_2}. \qquad (2.11) \end{aligned}$$

As $\varrho_{(\alpha,\beta,\gamma)}[f_i]_{g_1} < \varrho_{(\alpha,\beta,\gamma)}[f_j]_{g_1}$ and $\varrho_{(\alpha,\beta,\gamma)}[f_i]_{g_2} < \varrho_{(\alpha,\beta,\gamma)}[f_j]_{g_2}$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$, we obtain that

$$\text{either } \min\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\} > \min\{\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\} \text{ or}$$

$$\min\{\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\} > \min\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\} \text{ holds.}$$

Therefore in view of the conditions (i) and (ii) of the theorem, it follows from above that

$$\text{either } \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \pm g_2} > \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1 \pm g_2} \text{ or } \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1 \pm g_2} > \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \pm g_2}$$

which is the condition for holding equality in (2.11).

Hence the theorem follows. □

In line of Theorem 2.5 one can easily prove the following theorem with the help of Theorem 2.1 and Theorem 2.3, so we omit its proof.

Theorem 2.6. *Let f_1, f_2 be two meromorphic functions and g_1, g_2 be two entire functions such that the following conditions are satisfied:*

(i) $\lambda_{(\alpha,\beta,\gamma)}[f_i]_{g_1} > \lambda_{(\alpha,\beta,\gamma)}[f_j]_{g_1}$ with at least f_j is of regular relative (α,β,γ)-order with respect to g_1 for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and

(ii) $\lambda_{(\alpha,\beta,\gamma)}[f_i]_{g_2} > \lambda_{(\alpha,\beta,\gamma)}[f_j]_{g_2}$ *with at least* f_j *is of regular relative* (α,β,γ)*-order with respect to* g_2 *for* $i = 1,\ 2,\ j = 1,2$ *and* $i \neq j$.

$$\begin{aligned}&\lambda_{(\alpha,\beta,\gamma)}[f_1 \pm f_2]_{g_1 \pm g_2}\\ \geq\ &\min[\max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\}].\end{aligned}$$

The sign of equality holds when $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_i} < \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_j}$ *and* $\lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_i} < \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_j}$ *hold simultaneously for* $i = 1,2$; $j = 1,2$ *and* $i \neq j$.

Theorem 2.7. *Let* f_1, f_2 *be two meromorphic functions and* g_1 *be an entire function such that at least* f_1 *or* f_2 *is of regular relative* (α,β,γ)*-order with respect to* g_1*, then*

$$\lambda_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]_{g_1} \leq \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}.$$

The equality holds when $T_{f_1\cdot f_2}(r) > T_{f_1}(r)$ *and* $T_{f_1\cdot f_2}(r) > T_{f_2}(r)$.

Proof. Suppose that $\lambda_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]_{g_1} > 0$. Otherwise if $\lambda_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]_{g_1} = 0$ then the result is obvious. Let us consider that f_2 is of regular relative (α,β,γ)-order with respect to g_1. Also let that $\max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\} = \Delta$. We can clearly assume that $\lambda_{(\alpha,\beta,\gamma)}[f_k]_{g_1}$ is finite for $k = 1,2$. Now for any arbitrary $\varepsilon > 0$, it follows from the definition of $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}$, for a sequence values of r tending to infinity that

$$T_{f_1}(r) \leq T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1} + \varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right),$$

$$\text{(2.12)} \qquad i.e.,\ T_{f_1}(r) \leq T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\Delta + \varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right).$$

Also for any arbitrary $\varepsilon > 0$, we obtain from the definition of $\lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}$ $(= \rho_{(\alpha,\beta,\gamma)}[f_2]_{g_1})$, for all sufficiently large values of r that

$$T_{f_2}(r) \leq T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1} + \varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right),$$

$$\text{(2.13)} \qquad i.e.,\ T_{f_2}(r) \leq T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\Delta + \varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right).$$

Since $T_{f_1\cdot f_2}(r) \leq T_{f_1}(r) + T_{f_2}(r) + O(1)$ for all large values of r, so in view of (2.12) and (2.13), we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned}T_{f_1\cdot f_2}(r) &\leq 2\left[T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\Delta + \varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right)\right] + O(1),\\ i.e.,\ T_{f_1\cdot f_2}(r) &\leq 3\left[T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\Delta + \varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right)\right],\end{aligned}$$

$$
\begin{aligned}
i.e.,\ T_{f_1\cdot f_2}(r) &\leq \left[T_{g_1}\left(\exp^{[2]}\left(\alpha^{-1}\left((\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right)^3\right],\\
i.e.,\ T_{g_1}^{-1}T_{f_1\cdot f_2}(r) &\leq \left(\exp^{[2]}\left(\alpha^{-1}\left((\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right)\right)^3,\\
i.e.,\ \log T_{g_1}^{-1}T_{f_1\cdot f_2}(r) &\leq 3\exp\left(\alpha^{-1}\left((\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right)\right)\right),\\
i.e.,\ \log^{[2]} T_{g_1}^{-1}T_{f_1\cdot f_2}(r) &\leq \log 3+\alpha^{-1}\left((\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right)\right),
\end{aligned}
$$

$$
\begin{aligned}
i.e.,\ \log^{[2]} T_{g_1}^{-1}T_{f_1\cdot f_2}(r)-\log 3 &\leq \alpha^{-1}\left((\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right)\right),\\
i.e.,\ (1-o(1))\log^{[2]} T_{g_1}^{-1}T_{f_1\cdot f_2}(r) &\leq \alpha^{-1}\left((\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right)\right). \qquad (2.14)
\end{aligned}
$$

Since $\alpha \in L_2$, we have from (2.14), for a sequence of values of r tending to infinity that

$$
\begin{aligned}
i.e.,\ (1-o(1))\alpha\left(\log^{[2]} T_{g_1}^{-1}T_{f_1\cdot f_2}(r)\right) &\leq (\Delta+\varepsilon)\beta\left(\log(\gamma(r))\right),\\
i.e.,\ \frac{(1-o(1))\alpha\left(\log^{[2]} T_{g_1}^{-1}T_{f_1\cdot f_2}(r)\right)}{\beta\left(\log(\gamma(r))\right)} &\leq (\Delta+\varepsilon).
\end{aligned}
$$

Hence,

$$
\liminf_{r\to+\infty}\frac{(1-o(1))\alpha\left(\log^{[2]} T_{g_1}^{-1}T_{f_1\cdot f_2}(r)\right)}{\beta\left(\log(\gamma(r))\right)} \leq \Delta+\varepsilon,
$$

$$
i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1} \leq \Delta+\varepsilon.
$$

As $\varepsilon > 0$ is arbitrary,

$$
\lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1} \leq \Delta.
$$

Similarly, if we consider that f_1 is of regular relative (α,β,γ)-order with respect to g_1 or both f_1 and f_2 are of regular relative (α,β,γ)-order with respect to g_1, then also one can easily verify that

$$
\lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1} \leq \Delta.
$$

Hence,

$$
\lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1} \leq \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}. \qquad (2.15)
$$

This completes the first part of the theorem.

If $T_{f_1\cdot f_2}(r) > T_{f_1}(r)$ for all sufficiently large values of r, then

$$
T_{g_1}^{-1}(T_{f_1\cdot f_2}(r)) > T_{g_1}^{-1}(T_{f_1}(r)),
$$

as $T_{g_1}^{-1}(r)$ is an increasing function of r. Hence for all sufficiently large values of r,

$$\begin{aligned}\frac{\alpha(\log^{[2]} T_{g_1}^{-1}(T_{f_1\cdot f_2}(r)))}{\beta\left(\log(\gamma(r))\right)} &> \frac{\alpha(\log^{[2]} T_{g_1}^{-1}(T_{f_1}(r)))}{\beta\left(\log(\gamma(r))\right)},\\ i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1} &\geq \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}\end{aligned}$$

Similarly, $\lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1}\geq \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}$.Which implies that

$$\lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1}\geq \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1},\lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}. \tag{2.16}$$

In view of (2.15) and (2.16), we have

$$\lambda_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1}= \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1},\lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}.$$

This completes the proof. □

Now we state the following theorem which can easily be carried out in the line of Theorem 2.7 and therefore its proof is omitted.

Theorem 2.8. *Let f_1, f_2 be two meromorphic functions and g_1 be an entire function such that $\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}$, $\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}$ exist, then*

$$\varrho_{(\alpha,\beta,\gamma)}[f_1\cdot f_2]_{g_1}\leq \max\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1},\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}.$$

The equality holds when $T_{f_1\cdot f_2}(r)>T_{f_1}(r)$, $T_{f_1\cdot f_2}(r)>T_{f_2}(r)$.

Theorem 2.9. *Let f_1 be a meromorphic function and g_1, g_2 be two entire functions such that $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}$, $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}$ exist, then*

$$\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1\cdot g_2}\geq \min\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1},\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}.$$

The equality holds when $T_{g_1\cdot g_2}(r)>T_{g_1}(r)$, $T_{g_1\cdot g_2}(r)>T_{g_2}(r)$.

Proof. Suppose that $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1\cdot g_2}<\infty$. Otherwise if $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1\cdot g_2}=\infty$, then the result is obvious. Also let $\min\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1},\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}=\Delta$. We can clearly assume that $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_k}$ is finite for $k=1,2$. Now for any arbitrary $\varepsilon>0$, with $\varepsilon<\Delta$, we obtain for all sufficiently large values of r that

$$T_{g_k}(\exp^{[2]}\left(\alpha^{-1}[(\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_k}-\varepsilon)\beta\left(\log(\gamma(r))\right)]\right))\leq T_{f_1}(r),$$

$$i.e.,\ T_{g_k}(\exp^{[2]}\left(\alpha^{-1}[(\Delta-\varepsilon)\beta\left(\log(\gamma(r))\right)]\right))\leq T_{f_1}(r),$$

$$i.e.,\ T_{g_k}(r)\leq T_{f_1}\left[\gamma^{-1}\left(\exp\beta^{-1}\left(\frac{\alpha(\log^{[2]}r)}{\Delta-\varepsilon}\right)\right)\right]. \tag{2.17}$$

Since $T_{g_1 \cdot g_2}(r) \leq T_{g_1}(r) + T_{g_2}(r) + O(1)$ for all large values of r, so in view of (2.17) we have for all sufficiently large values of r that

$$T_{g_1 \cdot g_2}(r) \leq 2T_{f_1}\left[\left(\gamma^{-1}\left(\exp\beta^{-1}\left(\frac{\alpha(\log^{[2]} r)}{\Delta-\varepsilon}\right)\right)\right)\right] + O(1),$$

$$i.e.,\ T_{g_1 \cdot g_2}(r) \leq 3T_{f_1}\left[\left(\gamma^{-1}\left(\exp\beta^{-1}\left(\frac{\alpha(\log^{[2]} r)}{\Delta-\varepsilon}\right)\right)\right)\right],$$

$$i.e.,\ T_{g_1 \cdot g_2}(r) \leq T_{f_1}\left[\left(\gamma^{-1}\left(\exp\beta^{-1}\left(\frac{\alpha(\log^{[2]} r)}{\Delta-\varepsilon}\right)\right)\right)^3\right],$$

$$i.e.,\ T_{g_1 \cdot g_2}(\exp^{[2]}\left(\alpha^{-1}[(\Delta-\varepsilon)\beta\left(\log(\gamma(r^{\frac{1}{3}}))\right)]\right)) \leq T_{f_1}(r),$$

$$i.e.,\ (\exp^{[2]}\left(\alpha^{-1}[(\Delta-\varepsilon)\beta\left(\log(\gamma(r^{\frac{1}{3}}))\right)]\right)) \leq T^{-1}_{g_1 \cdot g_2}T_{f_1}(r). \tag{2.18}$$

Therefore in view of (2.18), it follows from above for all sufficiently large values of r that

$$\frac{\alpha(\log^{[2]} T^{-1}_{g_1 \cdot g_2}(T_{f_1}(r)))}{\beta\left(\log(\gamma(r^{\frac{1}{3}}))\right)} \geq \Delta - \varepsilon.$$

Hence,

$$\limsup_{r\to+\infty}\left[\frac{\alpha(\log^{[2]} T^{-1}_{g_1 \cdot g_2}(T_{f_1}(r)))}{\beta(\log(\gamma(r)))}\cdot\frac{\beta(\log(\gamma(r)))}{\beta\left(\log(\gamma(r^{\frac{1}{3}}))\right)}\right] \geq \Delta-\varepsilon,$$

$$\limsup_{r\to+\infty}\frac{\alpha(\log^{[2]} T^{-1}_{g_1 \cdot g_2}(T_{f_1}(r)))}{\beta(\log(\gamma(r)))}\cdot\lim_{r\to+\infty}\frac{\beta(\log(\gamma(r)))}{\beta\left(\log(\gamma(r^{\frac{1}{3}}))\right)} \geq \Delta-\varepsilon,$$

$$i.e.,\ \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \cdot g_2} \geq \Delta - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, therefore from above we get that

$$\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \cdot g_2} \geq \Delta = \min\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}. \tag{2.19}$$

Now, if $T_{g_1 \cdot g_2}(r) > T_{g_1}(r)$ for all sufficiently large values of r, then $T^{-1}_{g_1 \cdot g_2}(r) < T^{-1}_{g_1}(r)$. Hence

$$\frac{\alpha(\log T^{-1}_{g_1 \cdot g_2}(T_{f_1}(r)))}{\beta(\log(\gamma(r)))} < \frac{\alpha(\log T^{-1}_{g_1}(T_{f_1}(r)))}{\beta(\log(\gamma(r)))}.$$

So $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \cdot g_2} \leq \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}$. If $T_{g_1 \cdot g_2}(r) > T_{g_2}(r)$, similarly we get $\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \cdot g_2} \leq \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}$. Which implies that

$$\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \cdot g_2} \leq \min\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}. \tag{2.20}$$

In view of (2.19) and (2.20), we have

$$\lambda_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]_{g_1} = \min\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}.$$

Hence the theorem follows. □

Theorem 2.10. *Let f_1 be a meromorphic function and g_1, g_2 be two entire functions such that f_1 is of regular relative (α, β, γ)-order with respect to at least any one of g_1 or g_2, then*

$$\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1 \cdot g_2} \geq \min\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}.$$

The equality holds when $T_{g_1 \cdot g_2}(r) > T_{g_1}(r)$, $T_{g_1 \cdot g_2}(r) > T_{g_2}(r)$.

We omit the proof of Theorem 2.10 as it can easily be carried out in the line of Theorem 2.9.

Now we state the following two theorems without their proofs as those can easily be carried out in the line of Theorem 2.5 and Theorem 2.6 respectively.

Theorem 2.11. *Let f_1, f_2 be two meromorphic functions and g_1, g_2 be two entire functions such that*

(i) f_1 is of regular relative (α, β, γ)-order with respect to at least any one of g_1 or g_2,

(ii) f_2 is of regular relative (α, β, γ)-order with respect to at least any one of g_1 or g_2, and

(iii) $T_{g_1 \cdot g_2}(r) > T_{g_1}(r)$, $T_{g_1 \cdot g_2}(r) > T_{g_2}(r)$, then

$$\begin{aligned} & \varrho_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]_{g_1 \cdot g_2} \\ \leq \; & \max[\min\{\varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_1]_{g_2}\}, \min\{\varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_1}, \varrho_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\}]. \end{aligned}$$

The equality holds when $T_{f_1 \cdot f_2}(r) > T_{f_1}(r)$, $T_{f_1 \cdot f_2}(r) > T_{f_2}(r)$.

Theorem 2.12. *Let f_1, f_2 be two meromorphic functions and g_1, g_2 be two entire functions such that*

(i) At least f_1 or f_2 is of regular relative (α, β, γ)-order with respect to g_1,

(ii) At least f_1 or f_2 is of regular relative (α, β, γ)-order with respect to g_2, and

(iii) $T_{f_1 \cdot f_2}(r) > T_{f_1}(r)$, $T_{f_1 \cdot f_2}(r) > T_{f_2}(r)$, then

$$\begin{aligned} & \lambda_{(\alpha,\beta,\gamma)}[f_1 \cdot f_2]_{g_1 \cdot g_2} \\ \geq \; & \min[\max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_1}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_1}\}, \max\{\lambda_{(\alpha,\beta,\gamma)}[f_1]_{g_2}, \lambda_{(\alpha,\beta,\gamma)}[f_2]_{g_2}\}] \end{aligned}$$

The equality holds when $T_{g_1 \cdot g_2}(r) > T_{g_1}(r)$, $T_{g_1 \cdot g_2}(r) > T_{g_2}(r)$.

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