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NEW SUBCLASS OF p-VALENT CLOSE-TO-CONVEX FUNCTIONS

Gagandeep Singh [a,*] and Gurcharanjit Singh [b]

Abstract. In this paper, we introduce a new and generalized subclass of p-valently close-to-convex functions defined with subordination. We establish the coefficient estimates, inclusion relation, distortion theorem and argument theorem for this class. Some earlier known results follow as consequences.

1. Introduction

For $p \in \mathbb{N}$, let $\mathcal{A}_p$ denote the class of analytic functions f in the open unit disc $E = \{z : z \in \mathbb{C}, |z| < 1\}$ and has a Taylor series expansion of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n.$$

On putting $p = 1$ in $\mathcal{A}_p$, we obtain the class $\mathcal{A}_1$ of analytic functions of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. Further, $\mathcal{S}$ denote the class of functions in $\mathcal{A}_1$ which are univalent in E. A function w is said to be a Schwarz function if it has expansion of the form $w(z) = \sum_{n=1}^{\infty} c_n z^n$ and satisfy the conditions $w(0) = 0$ and $|w(z)| \leq 1$. The class of Schwarz functions is denoted by $\mathcal{U}$. An analytic function f is said to be subordinate to another analytic function g in E, if there exists a Schwarz function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$. If f is subordinate to g, then it is denoted by $f \prec g$. Moreover, if g is univalent in E, then $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(E) \subset g(E)$.

For $0 \leq \alpha < p$, the class of p-valently starlike functions of order α is denoted by $\mathcal{S}_p^*(\alpha)$ and is defined as

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[*]Corresponding author.

$$\mathcal{S}_p^*(\alpha)=\left\{f:f\in\mathcal{A}_p, Re\left(\frac{zf'(z)}{f(z)}\right)>\alpha, z\in E\right\}.$$

The class $\mathcal{S}_p^*(\alpha)$ was introduced by Goluzina [5]. For $0\leq\alpha<1$, $\mathcal{S}_1^*(\alpha)\equiv\mathcal{S}^*(\alpha)$, the class of starlike functions of order α. Also $\mathcal{S}_p^*(0)\equiv\mathcal{S}_p^*$, the class of p-valent starlike functions. Further $\mathcal{S}_1^*(0)\equiv\mathcal{S}^*$, the well known class of starlike functions.

The class $\mathcal{C}_p(\alpha)$ of p-valent close-to-convex functions was introduced by Umezawa [12] and defined as

$$\mathcal{C}_p(\alpha)=\left\{f:f\in\mathcal{A}_p, Re\left(\frac{zf'(z)}{g(z)}\right)>\alpha, g\in\mathcal{S}_p^*, z\in E\right\}.$$

For $p=1$, $\alpha=0$, the class $\mathcal{C}_p(\alpha)$ reduces to $\mathcal{C}$ which is the class of close-to-convex functions introduced by Kaplan [7].

The class of starlike functions with respect to symmetric points is denoted by $\mathcal{S}_s^*$ and is defined as

$$Re\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>0.$$

Sakaguchi [11] established the class $\mathcal{S}_s^*$. Clearly $\dfrac{f(z)-f(-z)}{2}$ is a starlike function [3] in E and so $\mathcal{S}_s^*$ is a subclass of the class $\mathcal{C}$ of close-to-convex functions.

Further, Gao and Zhou [4] established the class $\mathcal{K}_S$ which is given by

$$\mathcal{K}_s=\left\{f:f\in\mathcal{A}, Re\left(\frac{-z^2f'(z)}{g(z)g(-z)}\right)>0, g\in\mathcal{S}^*\left(\frac{1}{2}\right), z\in E\right\},$$

where $\mathcal{S}^*\left(\frac{1}{2}\right)$ is the class of starlike functions of order $\frac{1}{2}$.

Kowalczyk and Les-Bomba [8] extended the class $\mathcal{K}_S$ by introducing the class $\mathcal{K}_S(\gamma), (0\leq\gamma<1)$ which is mentioned below

$$\mathcal{K}_s(\gamma)=\left\{f:f\in\mathcal{A}, Re\left(\frac{-z^2f'(z)}{g(z)g(-z)}\right)>\gamma, g\in\mathcal{S}^*\left(\frac{1}{2}\right), z\in E\right\}.$$

For $\gamma=0$, the class $\mathcal{K}_S(\gamma)$ reduces to the class $\mathcal{K}_S$.

Janowski [6] introduced the class $\mathcal{P}(A,B)$ of analytic functions of the form $q(z)=1+\sum_{k=1}^{\infty}p_kz^k$ such that $q(z)\prec\dfrac{1+Az}{1+Bz}$ where $-1\leq B<A\leq 1$. Further, for $0\leq\alpha<p$, Aouf [1] established the class $\mathcal{P}(A,B;p;\alpha)$, which consists of analytic functions of the form $q(z)=p+\sum_{k=1}^{\infty}p_kz^k$ such that $q(z)\prec\dfrac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz}$. For $p=1,\alpha=0$, the class $\mathcal{P}(A,B;p;\alpha)$ reduces to $\mathcal{P}(A,B)$.

Motivated by the above mentioned work, now we introduce the following generalized subclass of $\mathcal{A}_p$.

Definition 1. Let $\mathcal{K}_s^p(A, B; \eta)$ denote the class of functions $f \in \mathcal{A}_p$ which satisfy the conditions,

$$\frac{-z^2 f'(z)}{z^{p-1} g(z) g(-z)} \prec \frac{p + [pB + (A - B)(p - \eta)]z}{1 + Bz}, -1 \leq B < A \leq 1, z \in E,$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \left(\frac{1}{2}\right)$, and $0 \leq \eta < p, -1 \leq B < A \leq 1$.

The following observations are obvious.
(i) For $\eta = 0, A = 1, B = -1$, the class $\mathcal{K}_s^p(A, B; \eta)$ agrees with $\mathcal{K}_s^p$, defined in [9].
(ii) On putting $\eta = 0, A = 1 - 2\gamma, B = -1$, the class $\mathcal{K}_s^p(A, B; \eta)$ reduces to $\mathcal{K}_s^p(\gamma)$, introduced in [9].
(iii) Taking $p = 1, \eta = 0, A = 1, B = -1$, the class $\mathcal{K}_s^p(A, B; \eta)$ gives the class $\mathcal{K}_s$ which was introduced by Gao and Zhou [4].
(iv) Substituting $p = 1, \eta = 0, A = 1 - 2\gamma, B = -1$ in $\mathcal{K}_s^p(A, B; \eta)$, the class $\mathcal{K}_s(\gamma)$ studied by Kowalczyk and Les Bomba [8], can be easily obtained.

By definition of subordination, it follows that $f \in \mathcal{K}_s^p(A, B; \eta)$ implies

$$\frac{-z^2 f'(z)}{z^{p-1} g(z) g(-z)} = \frac{p + [pB + (A - B)(p - \eta)]w(z)}{1 + Bw(z)} = P(z), w \in \mathcal{U}. \tag{1}$$

We establish various properties such as coefficient estimates, inclusion relationship, distortion theorem and argument theorem for the functions in the class $\mathcal{K}_s^p(A, B; \eta)$. For particular values of A, B and η, some earlier known results follow as special cases.

In the sequel, we assume that $-1 \leq B < A \leq 1, 0 \leq \eta < p, p \in \mathbb{N}, z \in E$.

2. Preliminary Results

For deriving the main results, we need the following lemmas:

Lemma 1 ([9]). *If*

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{S}^* \left(\frac{1}{2}\right),$$

then

$$G(z) = \frac{-g(z) g(-z)}{z} = z + \sum_{n=2}^{\infty} d_{2n-1} z^{2n-1}$$

is an odd starlike function and $|d_{2n-1}| \leq 1$, $n \in \mathbb{N} - \{1\}$.

Lemma 2 ([1]). *Let,*

$$\frac{p+[pB+(A-B)(p-\eta)]w(z)}{1+Bw(z)} = P(z) = p + \sum_{n=1}^{\infty} q_n z^n, \tag{2}$$

then

$$|q_n| \leq (p-\eta)(A-B), n \geq 1.$$

Lemma 3 ([10]). *Let* $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, *then*

$$\frac{1+A_1 z}{1+B_1 z} \prec \frac{1+A_2 z}{1+B_2 z}.$$

Lemma 4 ([4]). *For* $g \in \mathcal{S}^*\left(\frac{1}{2}\right)$,

$$G(z) = \frac{-g(z)g(-z)}{z}$$

is an odd starlike function, and so for $|z| = r, 0 < r < 1$, *we have*

$$\frac{r}{1+r^2} \leq |G(z)| \leq \frac{r}{1-r^2}.$$

3. MAIN RESULTS

Theorem 1. *If* $f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \in \mathcal{K}_s^p(A, B; \eta)$, *then for* n *odd, we have*

$$|a_{p+n}| \leq \frac{n+1}{2(p+n)}(A-B)(p-\eta) \tag{3}$$

and when n *is even*

$$|a_{p+n}| \leq \frac{1}{2(p+n)}[n(A-B)(p-\eta)+2p]. \tag{4}$$

Proof. As $f \in \mathcal{K}_s^p(A, B; \eta)$, therefore (1) yields

$$\frac{-z^2 f'(z)}{z^{p-1}g(z)g(-z)} = P(z),$$

which can be further expressed as

$$\frac{zf'(z)}{z^{p-1}G(z)} = P(z), \tag{5}$$

where

$$G(z) = \frac{-g(z)g(-z)}{z} = z + \sum_{n=2}^{\infty} d_{2n-1} z^{2n-1}. \tag{6}$$

Using the expansions of $f(z)$, $G(z)$ and $P(z)$ in (5) and equating the coefficients of z^{n+p}, it yields

$$(p+n)a_{p+n} = q_n + d_3q_{n-2} + d_5q_{n-4} + ... + d_{n-2}q_3 + d_nq_1 \tag{7}$$

and

$$(p+n)a_{p+n} = q_n + d_3q_{n-2} + d_5q_{n-4} + ... + d_{n-1}q_2 + d_{n+1}p \tag{8}$$

Applying triangle inequality and using Lemmas 1 and 2 in (7) and (8), the results (3) and (4) can be easily obtained. □

Substituting for $\eta = 0, A = 1, B = -1$ in Theorem 1, we can easily obtain the following result:

Corollary 1. *If $f \in \mathcal{K}_s^p$, then*

$$|a_{p+n}| \le \frac{p(n+1)}{p+n}.$$

Taking $\eta = 0, A = 1 - 2\gamma, B = -1$, Theorem 1 yields the following result:

Corollary 2. *If $f \in \mathcal{K}_s^p(\gamma)$, then*
for n odd, we have

$$|a_{p+n}| \le \frac{p(n+1)(1-\gamma)}{p+n},$$

for n even,

$$|a_{p+n}| \le \frac{p[n(1-\gamma)+1]}{p+n}.$$

For $p = 1, \eta = 0, A = 1, B = -1$, Theorem 1 gives the following result:

Corollary 3. *If $f \in \mathcal{K}_s$, then*

$$|a_{n+1}| \le 1.$$

On putting $p = 1, \eta = 0, A = 1 - 2\gamma, B = -1$ in Theorem 1, the following result is obvious:

Corollary 4. *If $f \in \mathcal{K}_s(\gamma)$, then for n odd, we have*

$$|a_{n+1}| \le (1-\gamma),$$

for n even,

$$|a_{n+1}| \leq \frac{n(1-\gamma)+1}{n+1}.$$

Theorem 2. *If $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \eta_2 \leq \eta_1 < 1$, then*

$$\mathcal{K}_s^p(A_1, B_1; \eta_1) \subset \mathcal{K}_s^p(A_2, B_2; \eta_2).$$

Proof. As $f \in \mathcal{K}_s^p(A_1, B_1; \eta_1)$, so

$$\frac{-z^2 f'(z)}{z^{p-1} g(z) g(-z)} \prec \frac{p + [pB_1 + (A_1 - B_1)(p - \eta_1)]z}{1 + B_1 z}.$$

As $-1 \leq B_2 = B_1 < A_1 \leq A_2 \leq 1$ and $0 \leq \eta_2 \leq \eta_1 < 1$, we have

$$-1 \leq B_1 + \frac{(p-\eta_1)(A_1 - B_1)}{p} \leq B_2 + \frac{(p-\eta_2)(A_2 - B_2)}{p} \leq 1.$$

Thus by Lemma 3, it yields

$$\frac{-z^2 f'(z)}{z^{p-1} g(z) g(-z)} \prec \frac{p + [pB_2 + (A_2 - B_2)(p - \eta_2)]z}{1 + B_2 z},$$

which implies $f \in \mathcal{K}_s^p(A_2, B_2; \eta_2)$. □

Theorem 3. If $f \in \mathcal{K}_s^p(A, B; \eta)$, then for $|z| = r, 0 < r < 1$, we have

$$\left(\frac{p - [pB + (A-B)(p-\eta)]r}{1 - Br}\right)\left(\frac{r^{p-1}}{1+r^2}\right) \leq |f'(z)|$$

$$\leq \left(\frac{p + [pB + (A-B)(p-\eta)]r}{1 + Br}\right)\left(\frac{r^{p-1}}{1-r^2}\right) \tag{9}$$

and

$$\int_0^r \left(\frac{p - [pB + (A-B)(p-\eta)]t}{1 - Bt}\right)\left(\frac{t^{p-1}}{1+t^2}\right) dt \leq |f(z)|$$

$$\leq \int_0^r \left(\frac{p + [pB + (A-B)(p-\eta)]t}{1 + Bt}\right)\left(\frac{t^{p-1}}{1-t^2}\right) dt. \tag{10}$$

Proof. (5) can be written as

$$|zf'(z)| = |z^{p-1} G(z)||P(z)|. \tag{11}$$

Aouf [2] proved that

$$\frac{p - [pB + (A-B)(p-\eta)]r}{1 - Br} \leq |P(z)| \leq \frac{p + [pB + (A-B)(p-\eta)]r}{1 + Br}. \tag{12}$$

Since G is an odd starlike function, so by Lemma 4, we have

$$\frac{r}{1+r^2} \leq |G(z)| \leq \frac{r}{1-r^2}. \tag{13}$$

Using (12) and (13) in (11), the result (9) can be easily obtained. On integrating (9) from 0 to r, (10) follows. □

Substituting for $p = 1, A = 1-2\gamma,\ B = -1,\ \eta = 0$ in Theorem 3, we can easily obtain the following result due to Kowalczyk and Les Bomba [8]:

Corollary 5. *If $f \in \mathcal{K}_s(\gamma)$, then for $|z| = r, 0 < r < 1$, we have*

$$\frac{1-(1-2\gamma)r}{(1+r)(1+r^2)} \leq |f'(z)| \leq \frac{1+(1-2\gamma)r}{(1-r)(1-r^2)}$$

and

$$\int_0^r \left(\frac{1-(1-2\gamma)t}{(1+t)(1+t^2)}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{1+(1-2\gamma)t}{(1-t)(1-t^2)}\right) dt.$$

Taking $p = 1,\ A = 1, B = -1,\ \eta = 0$, Theorem 3 yields the following result due to Gao and Zhou [4]:

Corollary 6. *If $f \in \mathcal{K}_s$, then for $|z| = r, 0 < r < 1$, we have*

$$\frac{1-r}{(1+r)(1+r^2)} \leq |f'(z)| \leq \frac{1+r}{(1-r)(1-r^2)}$$

and

$$\int_0^r \left(\frac{1-t}{(1+t)(1+t^2)}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{1+t}{(1-t)(1-t^2)}\right) dt.$$

On putting $\eta = 0, A = 1, B = -1$ in Theorem 3, the following result is obvious:

Corollary 7. *If $f \in \mathcal{K}_s^p$, then for $|z| = r, 0 < r < 1$, we have*

$$\left(\frac{p(1-r)}{1+r}\right)\left(\frac{r^{p-1}}{1+r^2}\right) \leq |f'(z)| \leq \left(\frac{p(1+r)}{1-r}\right)\left(\frac{r^{p-1}}{1-r^2}\right)$$

and

$$\int_0^r \left(\frac{p(1-t)}{1+t}\right)\left(\frac{t^{p-1}}{1+t^2}\right) dt \leq |f(z)| \leq \int_0^r \left(\frac{p(1+t)}{1-t}\right)\left(\frac{t^{p-1}}{1-t^2}\right) dt.$$

For $\eta = 0, A = 1-2\gamma, B = -1$, Theorem 3 gives the following result:

Corollary 8. *If* $f \in \mathcal{K}_s^p(\gamma)$, *then*

$$\frac{p[1-(1-2\gamma)r]}{1+r}\left(\frac{r^{p-1}}{1+r^2}\right) \leq |f'(z)| \leq \frac{p[1+(1-2\gamma)r]}{1-r}\left(\frac{r^{p-1}}{1-r^2}\right)$$

and

$$\int_0^r \frac{p[1-(1-2\gamma)t]}{1+t}\left(\frac{t^{p-1}}{1+t^2}\right)dt \leq |f(z)| \leq \int_0^r \frac{p[1+(1-2\gamma)t]}{1-t}\left(\frac{t^{p-1}}{1-t^2}\right)dt.$$

Theorem 4. *If* $f \in \mathcal{K}_s^p(A,B;\eta)$, *then for* $|z|=r, 0<r<1$, *we have*

$$\left|arg\frac{f'(z)}{z^{p-1}}\right| \leq 2sin^{-1}(r) + sin^{-1}\left(\frac{(A-B)(p-\eta)r}{p-[pB+(A-B)(p-\eta)]Br^2}\right). \tag{14}$$

Proof. (5) can be rewritten as

$$zf'(z) = z^{p-1}G(z)P(z),$$

which implies

$$\left|arg\frac{f'(z)}{z^{p-1}}\right| \leq |argP(z)| + \left|arg\frac{G(z)}{z}\right|. \tag{15}$$

It was proved by Aouf [1] that

$$|argP(z)| \leq sin^{-1}\left(\frac{(A-B)(p-\eta)r}{p-[pB+(A-B)(p-\eta)]Br^2}\right). \tag{16}$$

Also G is an odd starlike function, so it is well known that,

$$\left|arg\frac{G(z)}{z}\right| \leq 2sin^{-1}r. \tag{17}$$

Using (16) and (17) in (15), the result (14) can be easily obtained. □

REFERENCES

1. M.K. Aouf: On a class of p-valent starlike functions of order α. Int. J. Math. Math. Sci. **10** (1987), no. 4, 733-744. `https://doi.org/10.1155/S0161171287000838`
2. M.K. Aouf: On subclasses of p-valent close-to-convex functions. Tamkang J. Math. **22** (1991), no. 2, 133-143. `https://doi.org/10.5556/j.tkjm.22.1991.4586`
3. R.N. Das & P. Singh: On subclasses of schlicht mapping. Indian J. Pure Appl. Math. **8** (1977), no. 8, 864-872.
4. C.Y. Gao & S.Q. Zhou: On a class of analytic functions related to the starlike functions. Kyungpook Math. J. **45** (2005), 123-130.

5. E.G. Goluzina: On the coefficients of a class of functions, regular in a disk having an integral representation in it. J. Soviet Math. **2** (1974), no. 6, 606-617. https://doi.org/10.1007/BF01089939
6. W. Janowski: Some extremal problems for certain families of analytic functions I. Ann. Pol. Math. **28** (1973), 297-326. https://doi.org/10.4064/AP-28-3-297-326
7. W. Kaplan: Close-to-convex schlicht functions. Michigan Math. J. **1** (1952), 169-185. https://doi.org/10.1307/MMJ/1028988895
8. J. Kowalczyk & E. Les-Bomba: On a subclass of close-to-convex functions. Appl. Math. Letters **23** (2010), 1147-1151. https://doi.org/10.1016/j.aml.2010.03.004
9. W. Li & Q. Xu: On the properties of a certain subclass of close-to-convex functions. J. Appl. Anal. and Comput. **5** (2015), no. 4, 581-588. https://doi.org/10.11948/2015045
10. M.S. Liu: On a subclass of p-valent close-to-convex functions of order β and type α. J. Math. Study **30** (1997), 102-104.
11. K. Sakaguchi: On a certain univalent mapping. J. Math. Soc. Japan **11** (1959), 72-75. https://doi.org/10.2969/JMSJ/01110072
12. T. Umezawa: Multivalently close-to-convex functions. Proc. Amer. Math. Soc. **8** (1957), 869-874. https://doi.org/10.2307/2033681

[a]Professor: Department of Mathematics, Khalsa College, Amritsar, Punjab, India
Email address: kamboj.gagandeep@yahoo.in

[b]Professor: Department of Mathematics, G.N.D.U. College, Chungh, Tarn-Taran(Punjab), India
Email address: dhillongs82@yahoo.com