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# EULER'S LINE SEGMENT THEORY IN LORENTZIAN AND ISOTROPIC PLANE

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Abstract. We examine the length theory of line segments in the Lorentzian plane and the isotropic plane, with emphases the definition of distance.

## 1. Introduction

We became interested in the connection between the Euclidean plane $\mathbb{E}^2$, the Lorentzian plane $\mathbb{L}^2$ and the isotropic plane $\mathbb{I}^2$, while trying to find a way to relate Lorentzian geometry to isotropic geometry. We want to know if it is possible to successively transform Euclidean geometry into isotropic geometry and then into Lorentzian geometry in the definition of the dot product of geometry. An approach to connect the three planes is to consider $\mathbb{R}^2 \ni (x, y)$ having a metric series $ds_\epsilon^2 := dx^2 + \epsilon dy^2$ for $\epsilon \in \mathbb{R}$. If $\epsilon = 1$, $\epsilon = 0$, $\epsilon = -1$, we get $\mathbb{E}^2$, $\mathbb{I}^2$, $\mathbb{L}^2$, where the metric determines the geometry more or less completely. Euler did various studies on straight lines. We prove the theory presented by Euler in [1] using the definition of length. And we prove that the theory holds in the Lorentzian plane and the isotropic plane using the definition of length. The goal of this article is to apply the theorem of straight lines proposed by Euler to the Lorentzian plane $\mathbb{L}^2$ and the isotropic plane $\mathbb{I}^2$. We can see that the connectivity of the three planes by proving them using the metrics of the three planes. And this shows great possibility for future research connecting the three planes. We study the metric of the Lorentz plane in [8] and [9] and the metric of the isotropic plane in [7].

This article is organized as follows: In Section 2, we provide geometric interpretations of the distances in $\mathbb{E}^2$, $\mathbb{L}^2$ and $\mathbb{I}^2$. In Section 3, we study the properties of

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This thesis is based on the lectures of Professor Seong-Deog Yang from the Department of Mathematics at Korea University. I sincerely hope that this thesis serves as a modest way to repay Professor Yang's guidance.

triangle segment length ratios in $\mathbb{L}^2$ and $\mathbb{I}^2$ based on the properties of triangle segment length ratios in $\mathbb{E}^2$. In Section 4, we study the extension of the 'problem raised by Fermat in E135' considered by Euler by applying the definition of the circle of $\mathbb{L}^2$ and $\mathbb{I}^2$. In Section 5, we study that the linear separation formula holds in all three planes $\mathbb{E}^2$, $\mathbb{L}^2$ and $\mathbb{I}^2$.

## 2. Preliminaries

The Euclidean plane $\mathbb{E}^2$, the isotropic plane $\mathbb{I}^2$, and the Lorentzian plane $\mathbb{L}^2$ are related by the following equation for the metric:

$$ds^2 = dx^2 + \epsilon dy^2.$$

If $\epsilon = 1, 0, -1$, we obtain the metric of $\mathbb{E}^2, \mathbb{I}^2, \mathbb{L}^2$, respectively. The descriptations about Euclidean plane $\mathbb{E}^2$ and the Lorentzian plane $\mathbb{L}^2$ in this paper are based on the terminologies, notation, and contents of [3], [4], [5] and [6]. In particular, we use $x, y$ for the standard coordinate system for $\mathbb{E}^2$, $x, t$ for $\mathbb{L}^2$, and $x, l$ for $\mathbb{I}^2$. When expressed in $x, l$ coordinates, the metric of the isotropic plane is $dx^2$. The inner product of the isotropic plane is naturally defined, similar to $\mathbb{E}^2$ and $\mathbb{L}^2$.

**Definition 2.1.** (Euclidean plane) For two vectors $(x_1, y_1), (x_2, y_2) \in \mathbb{E}^2$, define

$$\langle (x_1, y_1), (x_2, y_2) \rangle_{\mathbb{E}} := x_1x_2 + y_1y_2. \tag{2.1}$$

$\langle\ ,\ \rangle_{\mathbb{E}}$ is called the *Euclidean inner product.*

(Lorentzian plane) For two vectors $(x_1, t_1), (x_2, t_2) \in \mathbb{L}^2$, define

$$\langle (x_1, t_1), (x_2, t_2) \rangle_{\mathbb{L}} := x_1x_2 - t_1t_2. \tag{2.2}$$

$\langle\ ,\ \rangle_{\mathbb{L}}$ is called the *Lorentzian inner product.*

(isotropic plane) For two vectors $(x_1, l_1), (x_2, l_2) \in \mathbb{I}^2$, define

$$\langle (x_1, l_1), (x_2, l_2) \rangle_{\mathbb{I}} := x_1x_2. \tag{2.3}$$

$\langle\ ,\ \rangle_{\mathbb{I}}$ is called the *isotropic inner product.*

**Definition 2.2.** (Euclidean plane) Given two vectors $X = (x_1, y_1)$ and $Y = (x_2, y_2)$, we define

$$d(X, Y)_{\mathbb{E}} := ||X - Y||_{\mathbb{E}} \tag{2.4}$$

which is called the *distance* between X and Y.

(Lorentzian plane) Given two vectors $X = (x_1, t_1)$ and $Y = (x_2, t_2)$, we define

$$d(X,Y)_{\mathbb{L}} := ||X - Y||_{\mathbb{L}} \tag{2.5}$$

which is called the *distance* between X and Y.

(isotropic plane) Given two vectors $X = (x_1, l_1)$ and $Y = (x_2, l_2)$, we define

$$d(X,Y)_{\mathbb{I}} := ||X - Y||_{\mathbb{I}} \tag{2.6}$$

which is called the *distance* between X and Y.

**Lemma 2.3.** *The distance between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ on the Euclidean plane is*

$$d(A,B)_{\mathbb{E}} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2}. \tag{2.7}$$

*The distance between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ on the Lorentzian plane is*

$$d(A,B)_{\mathbb{L}} = \sqrt{(b_1 - a_1)^2 - (b_2 - a_2)^2}. \tag{2.8}$$

*The distance between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ on the isotropic plane is*

$$d(A,B)_{\mathbb{I}} = \sqrt{(b_1 - a_1)^2}. \tag{2.9}$$

*Proof.* Vector norm in each plane is

$$||A - B||_{\mathbb{E}} = \sqrt{\langle (b_1 - a_1, b_2 - a_2), (b_1 - a_1, b_2 - a_2)\rangle_{\mathbb{E}}} = \sqrt{(b_1 - a_1)^2 + (b_2 - a_2)^2},$$

$$||A - B||_{\mathbb{L}} = \sqrt{\langle (b_1 - a_1, b_2 - a_2), (b_1 - a_1, b_2 - a_2)\rangle_{\mathbb{L}}} = \sqrt{(b_1 - a_1)^2 - (b_2 - a_2)^2},$$

$$||A - B||_{\mathbb{I}} = \sqrt{\langle (b_1 - a_1, b_2 - a_2), (b_1 - a_1, b_2 - a_2)\rangle_{\mathbb{I}}} = \sqrt{(b_1 - a_1)^2},$$

so the result holds. □

## 3. Specific Properties for Triangle Segment Length Ratios in $\mathbb{E}^2, \mathbb{L}^2, \mathbb{I}^2$

Euler proposed Specific properties for triangle segment length ratios in $\mathbb{E}^2$. In this section, we show whether the Specific properties for triangle segment length ratios suggested by Euler [2] holds in $\mathbb{L}^2, \mathbb{I}^2$. In particular, the distance formula between two points in $\mathbb{E}^2, \mathbb{L}^2, \mathbb{I}^2$ of (2.7), (2.8), (2.9) is used in the process of proving this theorem.

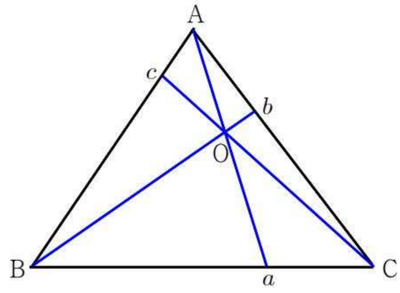


*Figure* 1. Triangle segment length ratios $\mathbb{E}^2$

**Theorem 3.1** (Euler's Theorem of 1780 in $\mathbb{E}^2$)**.** *In a triangle ABC with concurrent cevians Aa, Bb, and Cc as defined above with O their point of concurrency, the following property hold:*

$$\frac{AO_{\mathbb{E}}}{Oa_{\mathbb{E}}} \cdot \frac{BO_{\mathbb{E}}}{Ob_{\mathbb{E}}} \cdot \frac{CO_{\mathbb{E}}}{Oc_{\mathbb{E}}} = \frac{AO_{\mathbb{E}}}{Oa_{\mathbb{E}}} + \frac{BO_{\mathbb{E}}}{Ob_{\mathbb{E}}} + \frac{CO_{\mathbb{E}}}{Oc_{\mathbb{E}}} + 2. \tag{3.1}$$

*The notation for the length of line segment $AB$ in $\mathbb{E}^2$ is $AB_{\mathbb{E}}$ $(= d(A,B)_{\mathbb{E}})$.*

*Proof.* Let the vertices of triangle ABC be $A(a_1, a_2)$, $B(b_1, b_2)$, and $C(c_1, c_2)$, respectively. For convenience of calculation, let $b_1 = 0$ and $b_2 = 0$. The coordinate $a$ of the $m_1 : m_2$ internally dividing point of the line segment $BC$ is $a(\frac{m_1c_1}{m_1+m_2}, \frac{m_1c_2}{m_1+m_2})$, and the coordinate $b$ of the $n_1 : n_2$ internally dividing point of the line segment $CA$ is $b(\frac{n_1a_1+n_2c_1}{n_1+n_2}, \frac{n_1a_2+n_2c_2}{n_1+n_2})$. Therefore, the straight line passing through $A$, $a$ is

$$L_1 : y = \frac{(m_1+m_2)a_2 - m_1c_2}{(m_1+m_2)a_1 - m_1c_1}(x - a_1) + a_2,$$

and the straight line passing through $B$, $b$ is

$$L_2 : y = \frac{n_1a_2 + n_2c_2}{n_1a_1 + n_2c_1}x.$$

Finding the intersection $O$ of two straight lines $L_1$ and $L_2$ is

$$O(\frac{m_1(n_1a_1 + n_2c_1)}{m_1n_1 + m_1n_2 + m_2n_2}, \frac{m_1(n_1a_2 + n_2c_2)}{m_1n_1 + m_1n_2 + m_2n_2}).$$

Now, if the straight line passing through $A$ and $B$ is called $L$, then $L : y = \frac{a_2}{a_1}x$, and if the straight line passing through $O$ and $C$ is called $L_3$, then

$$L_3 : y = \frac{(m_1n_1 + m_2n_2)c_2 - m_1n_1a_2}{(m_1n_1 + m_2n_2)c_1 - m_1n_1a_1}(x - c_1) + c_2.$$

Finding the intersection point $c$ of two straight lines $L$ and $L_3$ is

$$c\left(\frac{m_1n_1}{m_1n_1+m_2n_2}a_1, \frac{m_1n_1}{m_1n_1+m_2n_2}a_2\right),$$

and $c$ is the internally dividing point of the line segment $AB$. Calculating by applying (2.7),

$$AO_{\mathbb{E}} = \frac{n_2\sqrt{((m_1+m_2)a_1 - m_1c_1)^2 + ((m_1+m_2)a_2 - m_1c_2)^2}}{m_1(n_1+n_2)+m_2n_2},$$
$$BO_{\mathbb{E}} = \frac{m_1\sqrt{(n_1a_1+n_2c_1)^2 + (n_1a_2+n_2c_2)^2}}{m_1(n_1+n_2)+m_2n_2},$$
$$CO_{\mathbb{E}} = \frac{\sqrt{(m_1n_1a_1 - (m_1n_1+m_2n_2)c_1)^2 + (m_1n_1a_2 - (m_1n_1+m_2n_2)c_2)^2}}{m_1(n_1+n_2)+m_2n_2}$$

and

$$Oa_{\mathbb{E}} = \frac{m_1n_1}{n_2(m_1+m_2)}AO_{\mathbb{E}}, \qquad Ob_{\mathbb{E}} = \frac{m_2n_2}{m_1(n_1+n_2)}BO_{\mathbb{E}},$$
$$Oc_{\mathbb{E}} = \frac{m_1n_2}{m_1n_1+m_2n_2}CO_{\mathbb{E}}.$$

Therefore, the result holds. □

Lets apply the content of Theorem 3.1 in the Lorentzian plane. We use the distance (2.8) of the Lorentzian plane.

**Theorem 3.2** (Euler's Theorem of 1780 in $\mathbb{L}^2$)**.** *In a triangle ABC with concurrent cevians Aa, Bb, and Cc as defined above with O their point of concurrency, the following property hold:*

$$\frac{AO_{\mathbb{L}}}{Oa_{\mathbb{L}}} \cdot \frac{BO_{\mathbb{L}}}{Ob_{\mathbb{L}}} \cdot \frac{CO_{\mathbb{L}}}{Oc_{\mathbb{L}}} = \frac{AO_{\mathbb{L}}}{Oa_{\mathbb{L}}} + \frac{BO_{\mathbb{L}}}{Ob_{\mathbb{L}}} + \frac{CO_{\mathbb{L}}}{Oc_{\mathbb{L}}} + 2. \tag{3.2}$$

*The notation for the length of line segment AB in $\mathbb{L}^2$ is $AB_{\mathbb{L}}$ $(= d(A,B)_{\mathbb{L}})$.*

*Proof.* Let the vertices of triangle ABC be $A(a_1,a_2)$, $B(b_1,b_2)$, and $C(c_1,c_2)$, respectively. For convenience of calculation, let $b_1 = 0$ and $b_2 = 0$. In the same way as the proof of Theorem 3.1, it is

$$a\left(\frac{m_1c_1}{m_1+m_2}, \frac{m_1c_2}{m_1+m_2}\right), \qquad b\left(\frac{n_1a_1+n_2c_1}{n_1+n_2}, \frac{n_1a_2+n_2c_2}{n_1+n_2}\right),$$
$$c\left(\frac{m_1n_1}{m_1n_1+m_2n_2}a_1, \frac{m_1n_1}{m_1n_1+m_2n_2}a_2\right),$$

and

$$O\left(\frac{m_1(n_1a_1+n_2c_1)}{m_1n_1+m_1n_2+m_2n_2}, \frac{m_1(n_1a_2+n_2c_2)}{m_1n_1+m_1n_2+m_2n_2}\right).$$

Calculating by applying (2.8),

$$AO_{\mathbb{L}} = \frac{n_2\sqrt{((m_1+m_2)a_1 - m_1c_1)^2 - ((m_1+m_2)a_2 - m_1c_2)^2}}{m_1(n_1+n_2)+m_2n_2},$$
$$BO_{\mathbb{L}} = \frac{m_1\sqrt{(n_1a_1+n_2c_1)^2 - (n_1a_2+n_2c_2)^2}}{m_1(n_1+n_2)+m_2n_2},$$
$$CO_{\mathbb{L}} = \frac{\sqrt{(m_1n_1a_1 - (m_1n_1+m_2n_2)c_1)^2 - (m_1n_1a_2 - (m_1n_1+m_2n_2)c_2)^2}}{m_1(n_1+n_2)+m_2n_2}$$

and

$$Oa_{\mathbb{L}} = \frac{m_1n_1}{n_2(m_1+m_2)}AO_{\mathbb{L}}, \qquad Ob_{\mathbb{L}} = \frac{m_2n_2}{m_1(n_1+n_2)}BO_{\mathbb{L}},$$
$$Oc_{\mathbb{L}} = \frac{m_1n_2}{m_1n_1+m_2n_2}CO_{\mathbb{L}}.$$

Therefore, the result holds. □

Finally, let us apply Theorem 3.1 to the isotropic plane. We use the distance (2.9) in the isotropic plane.

**Theorem 3.3** (Euler's Theorem of 1780 in $\mathbb{I}^2$)**.** *In a triangle ABC with concurrent cevians Aa, Bb, and Cc as defined above with O their point of concurrency, the following property hold:*

$$\frac{AO_{\mathbb{I}}}{Oa_{\mathbb{I}}} \cdot \frac{BO_{\mathbb{I}}}{Ob_{\mathbb{I}}} \cdot \frac{CO_{\mathbb{I}}}{Oc_{\mathbb{I}}} = \frac{AO_{\mathbb{I}}}{Oa_{\mathbb{I}}} + \frac{BO_{\mathbb{I}}}{Ob_{\mathbb{I}}} + \frac{CO_{\mathbb{I}}}{Oc_{\mathbb{I}}} + 2. \tag{3.3}$$

*The notation for the length of line segment AB in $\mathbb{I}^2$ is $AB_{\mathbb{I}}$ $(= d(A,B)_{\mathbb{I}})$.*

*Proof.* Let the vertices of triangle ABC be $A(a_1,a_2)$, $B(b_1,b_2)$, and $C(c_1,c_2)$, respectively. For convenience of calculation, let $b_1 = 0$ and $b_2 = 0$. In the same way as the proof of Theorem 3.1, it is

$$a(\frac{m_1c_1}{m_1+m_2}, \frac{m_1c_2}{m_1+m_2}), \qquad b(\frac{n_1a_1+n_2c_1}{n_1+n_2}, \frac{n_1a_2+n_2c_2}{n_1+n_2}),$$
$$c(\frac{m_1n_1}{m_1n_1+m_2n_2}a_1, \frac{m_1n_1}{m_1n_1+m_2n_2}a_2),$$

and

$$O(\frac{m_1(n_1a_1+n_2c_1)}{m_1n_1+m_1n_2+m_2n_2}, \frac{m_1(n_1a_2+n_2c_2)}{m_1n_1+m_1n_2+m_2n_2}).$$

Calculating by applying (2.9),

$$AO_{\mathbb{I}} = \frac{n_2\sqrt{((m_1+m_2)a_1 - m_1c_1)^2}}{m_1(n_1+n_2)+m_2n_2}, \qquad BO_{\mathbb{I}} = \frac{m_1\sqrt{(n_1a_1+n_2c_1)^2}}{m_1(n_1+n_2)+m_2n_2},$$
$$CO_{\mathbb{I}} = \frac{\sqrt{(m_1n_1a_1 - (m_1n_1+m_2n_2)c_1)^2}}{m_1(n_1+n_2)+m_2n_2}$$

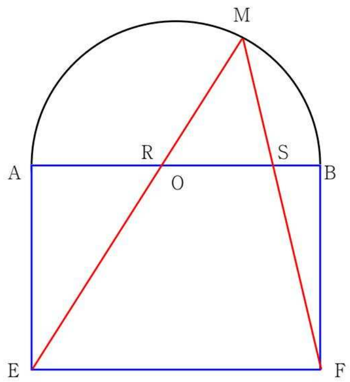


*Figure* 2. A Problem posed by Fermat in E135 in $\mathbb{E}^2$

and

$$Oa_{\mathbb{I}} = \frac{m_1 n_1}{n_2(m_1+m_2)} AO_{\mathbb{I}}, \qquad Ob_{\mathbb{I}} = \frac{m_2 n_2}{m_1(n_1+n_2)} BO_{\mathbb{I}},$$
$$Oc_{\mathbb{I}} = \frac{m_1 n_2}{m_1 n_1 + m_2 n_2} CO_{\mathbb{I}}.$$

Therefore, the result holds. □

In the Euclidean plane, Lorentzian plane, and isotropic plane,

$$\frac{AO}{Oa} \cdot \frac{BO}{Ob} \cdot \frac{CO}{Oc} = \frac{AO}{Oa} + \frac{BO}{Ob} + \frac{CO}{Oc} + 2$$

holds.

## 4. A Problem posed by Fermat in E135 in $\mathbb{E}^2, \mathbb{L}^2, \mathbb{I}^2$

Euler considered a problem posed by Fermat in E135. In this section, we prove the problem (theorem) raised by Fermat in E135 and consider this problem (theorem) in $\mathbb{L}^2$ and $\mathbb{I}^2$. In particular, the Lorentz plane is divided into ‘spacelike plane’, ‘timelike plane’, and ‘lightlike plane’. And the equations of the circle in the three planes are also different. We prove that the theorem holds in ‘spacelike plane’ and ‘timelike plane’ respectively. In ‘lightlike plane’, circles are excluded because they are straight lines. On the other hand, there are two equations of a circle in the isotropic plane. We prove that the theorem holds as a circle whose center is a point.

**Theorem 4.1** (Semicircle in $\mathbb{E}^2$)**.** *There is a semicircle with a rectangle $ABFE$ erected on its diameter $AB$ as shown in Figure 2, where $AE_{\mathbb{E}}$ is $\frac{AB_{\mathbb{E}}}{\sqrt{2}}$ in length.*

*Form segments AE and AF that intersect AB in points R and S respectively. Then*

$$(AS_{\mathbb{E}})^2 + (BR_{\mathbb{E}})^2 = (AB_{\mathbb{E}})^2. \tag{4.1}$$

*Proof.* For convenience of calculation, let $A(-r, 0)$ and $B(r, 0)$. The center $O$ of a semicircle is $O(0, 0)$, and the formula of a semicircle is $x^2 + y^2 = r^2(y \geq 0)$. $AE_{\mathbb{E}}$ is $\frac{AB_{\mathbb{E}}}{\sqrt{2}}$, so it is $E(-r, -\sqrt{2}r)$ and $F(r, -\sqrt{2}r)$. Let $M(x_1, y_1)$ be any point $M$ on the semicircle $x^2 + y^2 = r^2(y \geq 0)$. The straight line passing through $M$ and $E$ is

$$L_1 : y = \frac{y_1 + \sqrt{2}r}{x_1 + r}(x + r) - \sqrt{2}r. \tag{4.2}$$

In the linear equation (4.2), if $y = 0$, then $R(\frac{\sqrt{2}x_1 - y_1}{y_1 + \sqrt{2}r} r, 0)$. The straight line passing through $M$ and $F$ is

$$L_2 : y = \frac{y_1 + \sqrt{2}r}{x_1 - r}(x - r) - \sqrt{2}r. \tag{4.3}$$

In the linear equation (4.3), if $y = 0$, then $S(\frac{\sqrt{2}x_1 + y_1}{y_1 + \sqrt{2}r} r, 0)$. Calculating by applying (2.7),

$$AS_{\mathbb{E}} = \frac{\sqrt{2}x_1 + y_1}{y_1 + \sqrt{2}r} r + r, \qquad BR_{\mathbb{E}} = r - \frac{\sqrt{2}x_1 - y_1}{y_1 + \sqrt{2}r} r, \qquad AB_{\mathbb{E}} = 2r.$$

Therefore, the result of (4.1) holds. □

We can see from the above proof that the points of $R$ and $S$ exist inside the line segment $AB$.

Lets apply the content of Theorem 4.1 in the Lorentzian plane. There are three types of circles in the Lorentzian plane. The equation of a circle with the center at $(x_0, t_0)$ is $\sqrt{(x - x_0)^2 - (t - t_0)^2} = r$, and $r \in \mathbb{R}^+ \cup 0 \cup i\mathbb{R}^+$. At this time, if $r \in \mathbb{R}^+$, it is a real circle, if $r = 0$, it is a zero circle, and if $r \in i\mathbb{R}^+$, it is an imaginary circle. At this time, the equation of the zero circle is $(x - x_0)^2 = (t - t_0)^2$. We exclude from the proof the zero circle, which appears as two perpendicular lines.

**Theorem 4.2** (Real semicircle in $\mathbb{L}^2$)**.** *There is a real semicircle with a rectangle ABFE erected on its diameter AB as shown in Figure 3, where* $AE_{\mathbb{L}}$ *is* $\frac{AB_{\mathbb{L}}}{\sqrt{2}}$ *in length. Form segments AE and AF that intersect AB in points R and S respectively. Then*

$$(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 \geq (AB_{\mathbb{L}})^2. \tag{4.4}$$

*Proof.* Let us first prove the case where the real semicircle is $x^2 - t^2 = r^2(r^2 > 0, t \geq 0)$. For convenience of calculation, let $A(-r, 0)$ and $B(r, 0)$. The center $O$

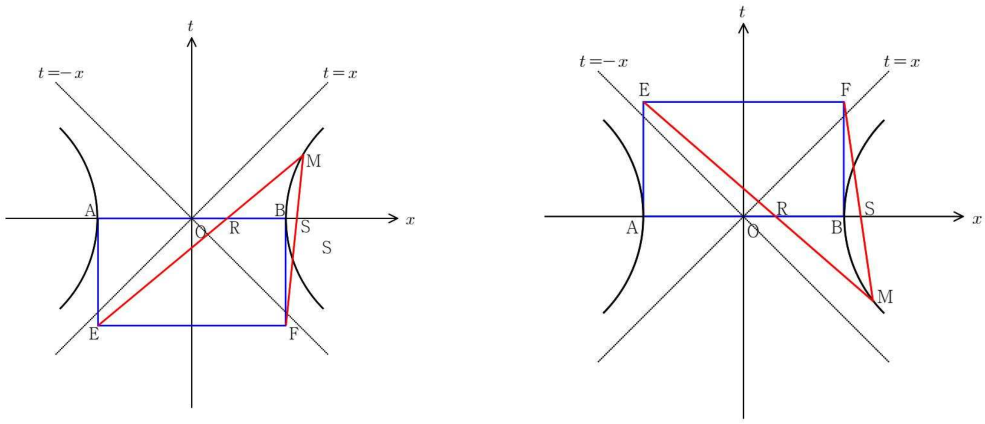


*Figure* 3. A Problem posed by Fermat in E135 in $\mathbb{L}^2$(spacelike)

of a real semicircle($x^2 - t^2 = r^2(r^2 > 0, t \geq 0)$) is $O(0,0)$. $AE_{\mathbb{L}}$ is $\frac{AB_{\mathbb{L}}}{\sqrt{2}}$, so it is $E(-r, -\sqrt{2}r)$ and $F(r, -\sqrt{2}r)$. Let $M(x_1, t_1)$ be any point $M$ on the real semicircle $x^2 - t^2 = r^2(r^2 > 0, t \geq 0)$. The straight line passing through $M$ and $E$ is

$$L_1 : t = \frac{t_1 + \sqrt{2}r}{x_1 + r}(x + r) - \sqrt{2}r. \tag{4.5}$$

In the linear equation (4.5), if $t = 0$, then $R(\frac{\sqrt{2}x_1 - t_1}{t_1 + \sqrt{2}r}r, 0)$. The straight line passing through $M$ and $F$ is

$$L_2 : t = \frac{t_1 + \sqrt{2}r}{x_1 - r}(x - r) - \sqrt{2}r. \tag{4.6}$$

In the linear equation (4.6), if $t = 0$, then $S(\frac{\sqrt{2}x_1 + t_1}{t_1 + \sqrt{2}r}r, 0)$. Calculating by applying (2.8),

$$AS_{\mathbb{L}} = \frac{\sqrt{2}x_1 + t_1}{t_1 + \sqrt{2}r}r + r, \qquad BR_{\mathbb{L}} = r - \frac{\sqrt{2}x_1 - t_1}{t_1 + \sqrt{2}r}r, \qquad AB_{\mathbb{L}} = 2r,$$

$$(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = 4r^2(1 + \frac{2{t_1}^2}{(t_1 + \sqrt{2}r)^2}). \tag{4.7}$$

If $1 + \frac{2{t_1}^2}{(t_1+\sqrt{2}r)^2} = 1$ in the equation (4.7), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = (AB_{\mathbb{L}})^2$. At this time, $x_1 = \pm r$, $t_1 = 0$. If $1 + \frac{2{t_1}^2}{(t_1+\sqrt{2}r)^2} > 1$ in the equation (4.7), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 > (AB_{\mathbb{L}})^2$. At this time, $x_1$,$t_1$ are all points that satisfy $x^2 - t^2 = r^2(t > 0)$. If $1 + \frac{2{t_1}^2}{(t_1+\sqrt{2}r)^2} < 1$ in the equation (4.7), the relation $(AS_{\mathbb{L}})^2 +$

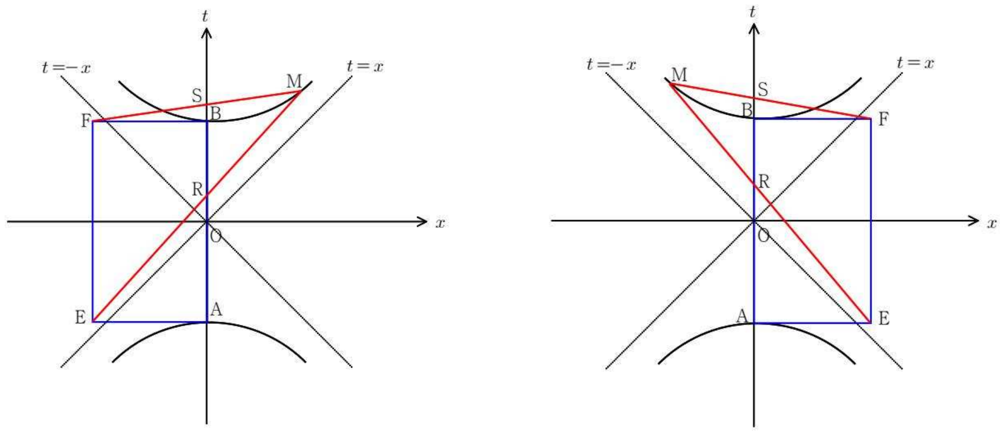


*Figure* 4. A Problem posed by Fermat in E135 in $\mathbb{L}^2$(timelike)

$(BR_{\mathbb{L}})^2 < (AB_{\mathbb{L}})^2$. At this time, $x_1$ and $t_1$ do not exist. Therefore, the result of (4.4) holds.

Now, let us prove the case where the real semicircle is $x^2 - t^2 = r^2 (r^2 > 0, t \le 0)$. Let $A(-r, 0)$, $B(r, 0)$, $E(-r, \sqrt{2}r)$, $F(r, \sqrt{2}r)$ and let $M(x_1, t_1)$ be any point $M$ on the real semicircle $x^2 - t^2 = r^2 (r^2 > 0, t \le 0)$. The straight line passing through $M$ and $E$ is

$$L_1 : t = \frac{t_1 - \sqrt{2}r}{x_1 + r}(x + r) + \sqrt{2}r. \tag{4.8}$$

In the linear equation (4.8), if $t = 0$, then $R(\frac{-\sqrt{2}x_1 - t_1}{t_1 - \sqrt{2}r} r, 0)$. The straight line passing through $M$ and $F$ is

$$L_2 : t = \frac{t_1 - \sqrt{2}r}{x_1 - r}(x - r) + \sqrt{2}r. \tag{4.9}$$

In the linear equation (4.9), if $t = 0$, then $S(\frac{-\sqrt{2}x_1 + t_1}{t_1 - \sqrt{2}r} r, 0)$. Therefore, $AS_{\mathbb{L}} = \frac{-\sqrt{2}x_1 + t_1}{t_1 - \sqrt{2}r} r + r$, $BR_{\mathbb{L}} = r - \frac{-\sqrt{2}x_1 - t_1}{t_1 - \sqrt{2}r} r$, $AB_{\mathbb{L}} = 2r$, and

$$(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = 4r^2 (1 + \frac{2{t_1}^2}{(t_1 - \sqrt{2}r)^2}). \tag{4.10}$$

If $1 + \frac{2{t_1}^2}{(t_1 - \sqrt{2}r)^2} = 1$ in the equation (4.10), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = (AB_{\mathbb{L}})^2$. At this time, $x_1 = \pm r$, $t_1 = 0$. If $1 + \frac{2{t_1}^2}{(t_1 - \sqrt{2}r)^2} > 1$ in the equation (4.10), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 > (AB_{\mathbb{L}})^2$. At this time, $x_1$,$t_1$ are all points that satisfy

$x^2 - t^2 = r^2 (t > 0)$. If $1 + \frac{2{t_1}^2}{(t_1 - \sqrt{2}r)^2} < 1$ in the equation (4.10), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 < (AB_{\mathbb{L}})^2$. At this time, $x_1$ and $t_1$ do not exist. Therefore, the result of (4.4) holds. □

**Theorem 4.3** (Imaginary semicircle in $\mathbb{L}^2$)**.** *There is a imaginary semicircle with a rectangle ABFE erected on its diameter AB as shown in Figure 4, where $AE_{\mathbb{L}}$ is $\frac{AB_{\mathbb{L}}}{\sqrt{2}}$ in length. Form segments AE and AF that intersect AB in points R and S respectively. Then*

$$(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 \geq (AB_{\mathbb{L}})^2. \tag{4.11}$$

*Proof.* Let us first prove the case where the imaginary semicircle is $x^2 - t^2 = -r^2 (r^2 > 0, x \geq 0)$. For convenience of calculation, let $A(0, -r)$ and $B(0, r)$. The center $O$ of a real semicircle($x^2 - t^2 = -r^2 (r^2 > 0, t \geq 0)$) is $O(0, 0)$. $AE_{\mathbb{L}}$ is $\frac{AB_{\mathbb{L}}}{\sqrt{2}}$, so it is $E(-\sqrt{2}r, -r)$ and $F(-\sqrt{2}r, r)$. Let $M(x_1, t_1)$ be any point $M$ on the real semicircle $x^2 - t^2 = -r^2 (r^2 > 0, t \geq 0)$. The straight line passing through $M$ and $E$ is

$$L_1 : t = \frac{-t_1 - r}{-x_1 - \sqrt{2}r}(x + \sqrt{2}r) - r. \tag{4.12}$$

In the linear equation (4.12), if $x = 0$, then $R(0, \frac{\sqrt{2}t_1 - x_1}{x_1 + \sqrt{2}r} r)$. The straight line passing through $M$ and $F$ is

$$L_2 : t = \frac{-t_1 + r}{-x_1 - \sqrt{2}r}(x + \sqrt{2}r) + r. \tag{4.13}$$

In the linear equation (4.13), if $x = 0$, then $S(0, \frac{\sqrt{2}t_1 + x_1}{x_1 + \sqrt{2}r} r)$. Calculating by applying (2.8),

$$AS_{\mathbb{L}} = (\frac{\sqrt{2}t_1 + x_1}{x_1 + \sqrt{2}r} r + r)i, \qquad BR_{\mathbb{L}} = (r - \frac{\sqrt{2}t_1 - x_1}{x_1 + \sqrt{2}r} r)i, \qquad AB_{\mathbb{L}} = (2r)i.$$

$$(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = -4r^2 (1 + \frac{2{x_1}^2}{(x_1 + \sqrt{2}r)^2}) \tag{4.14}$$

If $1 + \frac{2{x_1}^2}{(x_1 + \sqrt{2}r)^2} = 1$ in the equation (4.14), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = (AB_{\mathbb{L}})^2$. At this time, $t_1 = \pm r$, $x_1 = 0$. If $1 + \frac{2{x_1}^2}{(x_1 + \sqrt{2}r)^2)^2} > 1$ in the equation (4.14), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 > (AB_{\mathbb{L}})^2$. At this time, $x_1$,$t_1$ are all points that satisfy $x^2 - t^2 = r^2 (x > 0)$. If $1 + \frac{2{x_1}^2}{(x_1 + \sqrt{2}r)^2} < 1$ in the equation (4.14), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 < (AB_{\mathbb{L}})^2$. At this time, $x_1$ and $t_1$ do not exist. Therefore, the result of (4.11) holds.

Now, let us prove the case where the imaginary semicircle is $x^2 - t^2 = -r^2 (r^2 > 0, x \le 0)$. Let $A(0, -r)$, $B(0, r)$, $E(\sqrt{2}r, -r)$, $F(\sqrt{2}r, r)$ and let $M(x_1, t_1)$ be any point $M$ on the imaginary semicircle $x^2 - t^2 = -r^2 (r^2 > 0, x \le 0)$. The straight line passing through $M$ and $E$ is

$$L_1 : t = \frac{-t_1 - r}{-x_1 + \sqrt{2}r}(x - \sqrt{2}r) - r. \tag{4.15}$$

In the linear equation (4.15), if $x = 0$, then $R(0, \frac{\sqrt{2}t_1 + x_1}{-x_1 + \sqrt{2}r} r)$. The straight line passing through $M$ and $F$ is

$$L_2 : t = \frac{-t_1 + r}{-x_1 + \sqrt{2}r}(x - \sqrt{2}r) + r. \tag{4.16}$$

In the linear equation (4.16), if $x = 0$, then $S(0, \frac{\sqrt{2}t_1 - x_1}{-x_1 + \sqrt{2}r} r)$. Therefore, $AS_{\mathbb{L}} = (\frac{\sqrt{2}t_1 - x_1}{-x_1 + \sqrt{2}r} r + r)i$, $BR_{\mathbb{L}} = (r - \frac{\sqrt{2}t_1 + x_1}{-x_1 + \sqrt{2}r} r)i$, $AB_{\mathbb{L}} = (2r)i$, and

$$(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = -4r^2 (1 + \frac{2{x_1}^2}{(-x_1 + \sqrt{2}r)^2}). \tag{4.17}$$

If $1 + \frac{2{x_1}^2}{(-x_1 + \sqrt{2}r)^2} = 1$ in the equation (4.17), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 = (AB_{\mathbb{L}})^2$. At this time, $x_1 = \pm r$, $t_1 = 0$. If $1 + \frac{2{x_1}^2}{(-x_1 + \sqrt{2}r)^2} > 1$ in the equation (4.17), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 > (AB_{\mathbb{L}})^2$. At this time, $x_1$,$t_1$ are all points that satisfy $x^2 - t^2 = r^2 (t > 0)$. If $1 + \frac{2{x_1}^2}{(-x_1 + \sqrt{2}r)^2} < 1$ in the equation (4.17), the relation $(AS_{\mathbb{L}})^2 + (BR_{\mathbb{L}})^2 < (AB_{\mathbb{L}})^2$. At this time, $x_1$ and $t_1$ do not exist. Therefore, the result of (4.11) holds. □

We can see from the proofs of Theorem 4.2 and Theorem 4.3 that one of the points $R$ and $S$ is inside the line segment $AB$, and the other is outside the line segment $AB$. This is different from the Euclidean plane where both points $R$ and $S$ are inside the line segment $AB$.

Let us apply the content of Theorem 4.1 in the isotropic plane. There are two types of circles in the isotropic plane. The equation of a circle in an isotropic plane is $x^2 + Ax + By + C = 0$, and if $B = 0$, the center of the circle exists. We apply Theorem 4.1 to a circle with a center.

**Theorem 4.4** (Semicircle in $\mathbb{I}^2$). *There is a semicircle with a rectangle $ABFE$ erected on its diameter $AB$ as shown in Figure 5, where $AE_{\mathbb{I}}$ is $\frac{AB_{\mathbb{I}}}{\sqrt{2}}$ in length. Form segments $AE$ and $AF$ that intersect $AB$ in points $R$ and $S$ respectively. Then*

$$(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 \ge (AB_{\mathbb{I}})^2. \tag{4.18}$$

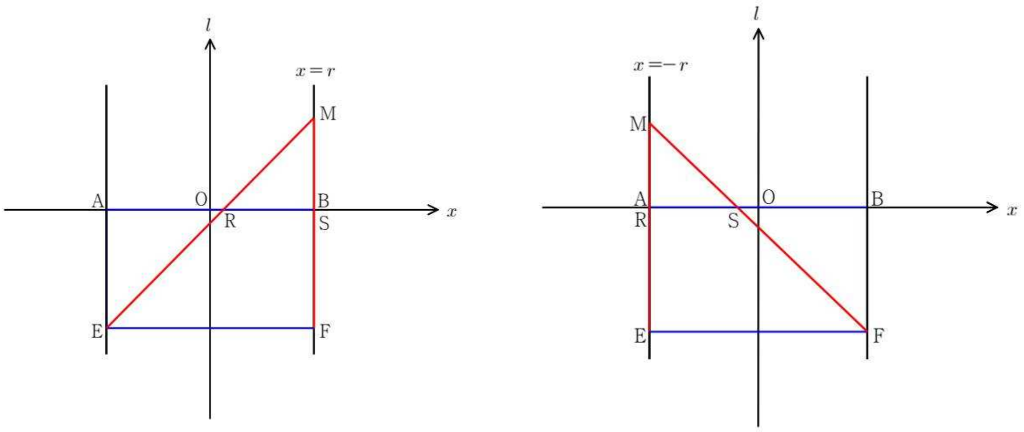


*Figure* 5. A Problem posed by Fermat in E135 in $\mathbb{I}^2$

*Proof.* Let the semicircle be $x^2 = r^2 (l \geq 0)$. For convenience of calculation, let $A(-r, 0)$ and $B(r, 0)$. The center $O$ of a semicircle($x^2 = r^2 (l \geq 0)$) is $O(0, 0)$. $AE_{\mathbb{I}}$ is $\frac{AB_{\mathbb{I}}}{\sqrt{2}}$, so it is $E(-r, -\sqrt{2}r)$ and $F(r, -\sqrt{2}r)$. First, let $M(r, l_1)$ be any point $M$ on the real semicircle $x = r (r > 0)$. The straight line passing through $M$ and $E$ is

$$L_1 : l = \frac{l_1 + \sqrt{2}r}{2r}(x + r) - \sqrt{2}r. \tag{4.19}$$

In the linear equation (4.19), if $l = 0$, then $R(\frac{\sqrt{2}r - l_1}{l_1 + \sqrt{2}r} r, 0)$. The point $S$ where the straight line passing through point $M$ and point $F$ and the $X - axis$ meet is $S(r, 0)$. Calculating by applying (2.9),

$$AS_{\mathbb{I}} = 2r, \qquad BR_{\mathbb{I}} = r - \frac{\sqrt{2}r - l_1}{l_1 + \sqrt{2}r} r, \qquad AB_{\mathbb{I}} = 2r.$$

$$(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 = 4r^2 (1 + \frac{{l_1}^2}{(l_1 + \sqrt{2}r)^2}) \tag{4.20}$$

If $1 + \frac{{l_1}^2}{(l_1 + \sqrt{2}r)^2} = 1$ in the equation (4.20), the relation $(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 = (AB_{\mathbb{I}})^2$. At this time, $x_1 = r$, $l_1 = 0$. If $1 + \frac{{l_1}^2}{(l_1 + \sqrt{2}r)^2} > 1$ in the equation (4.20), the relation $(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 > (AB_{\mathbb{I}})^2$. At this time, $x_1$,$l_1$ are all points that satisfy $x = r (r > 0)$. If $1 + \frac{{l_1}^2}{(l_1 + \sqrt{2}r)^2} < 1$ in the equation (4.20), the relation $(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 < (AB_{\mathbb{I}})^2$. At this time, $x_1$ and $l_1$ do not exist. Therefore, the result of (4.18) holds.

Now, let us prove the case where the real semicircle is $x = -r (r > 0)$. Let $A(-r, 0)$, $B(r, 0)$, $E(-r, -\sqrt{2}r)$, $F(r, -\sqrt{2}r)$ and let $M(x_1, l_1)$ be any point $M$ on the real semicircle $x = -r(r > 0)$. The straight line passing through $M$ and $F$ is

$$L_2 : l = \frac{-l_1 - \sqrt{2}r}{2r}(x - r) - \sqrt{2}r. \tag{4.21}$$

In the linear equation (4.21), if $l = 0$, then $S(\frac{-\sqrt{2}x_1 - t_1}{t_1 - \sqrt{2}r}r, 0)$. The point $R$ where the straight line passing through point $M$ and point $E$ and the $X - axis$ meet is $R(-r, 0)$. Therefore, $AS_{\mathbb{I}} = \frac{l_1 - \sqrt{2}r}{l_1 + \sqrt{2}r}r + r$, $BR_{\mathbb{I}} = 2r$, $AB_{\mathbb{I}} = 2r$, and

$$(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 = 4r^2(1 + \frac{{l_1}^2}{(l_1 + \sqrt{2}r)^2}) \tag{4.22}$$

If $1 + \frac{{l_1}^2}{(l_1+\sqrt{2}r)^2} = 1$ in the equation (4.22), the relation $(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 = (AB_{\mathbb{I}})^2$. At this time, $x_1 = -r$, $l_1 = 0$. If $1+\frac{{l_1}^2}{(l_1+\sqrt{2}r)^2} > 1$ in the equation (4.22), the relation $(AS_{\mathbb{I}})^2 + (BR_{\mathbb{I}})^2 > (AB_{\mathbb{I}})^2$. At this time, $x_1$,$l_1$ are all points that satisfy $x = -r(r > 0)$. If $1+\frac{{l_1}^2}{(l_1+\sqrt{2}r)^2} < 1$ in the equation (4.22), the relation $(AS_{\mathbb{I}})^2+(BR_{\mathbb{I}})^2 < (AB_{\mathbb{I}})^2$. At this time, $x_1$ and $l_1$ do not exist. Therefore, the result of (4.18) holds. It is proven in the same way for the semicircle $x^2 = r^2(l \leq 0)$. □

We can see from the proof of Theorem 4.4 that one of the points $R$ and $S$ exists inside the line segment $AB$, and the other point is equal to point $A$ or $B$. Therefore, it can be seen that the positions of points $R$ and $S$ continuously change in the Euclidean plane, isotropic plane, and Lorentzian plane.

## 5. LINEAR SEPARATION IN $\mathbb{E}^2, \mathbb{L}^2, \mathbb{I}^2$

For the problem proved in section 4, Euler shows a lemma. In this section, we prove this lemma on $\mathbb{E}^2$, $\mathbb{L}^2$, and $\mathbb{I}^2$. There are several proofs of this lemma, but we prove it using (2.7), (2.8), (2.9).

**Lemma 5.1.** *For any collinear points $X$, $Y$, $Z$, $W$ in the Euclidean plane are given in that order along their line,*

$$XW_{\mathbb{E}} \cdot YZ_{\mathbb{E}} + XY_{\mathbb{E}} \cdot WZ_{\mathbb{E}} = XZ_{\mathbb{E}} \cdot WY_{\mathbb{E}}. \tag{5.1}$$

*Proof.* Let $y = mx + n$ be the straight line passing through the points $X$, $Y$, $Z$, and $W$. For $x_1 < y_1 < z_1 < w_1$, let point $X$ be $X(x_1, mx_1+n)$, point $Y$ be $Y(y_1, my_1+n)$, point $Z$ be $Z(z_1, mz_1 + n)$, and point $W$ be $W(w_1, mw_1 + n)$. Applying (2.7) to calculate the distance between two points gives us

$$XW_{\mathbb{E}} = \sqrt{1 + m^2}\,|w_1 - x_1|, \qquad YZ_{\mathbb{E}} = \sqrt{1 + m^2}\,|z_1 - y_1|,$$

$$XY_{\mathbb{E}} = \sqrt{1+m^2}\,|y_1 - x_1|\,, \qquad WZ_{\mathbb{E}} = \sqrt{1+m^2}\,|z_1 - w_1|\,,$$

and

$$XZ_{\mathbb{E}} = \sqrt{1+m^2}\,|z_1 - x_1|\,, \qquad WY_{\mathbb{E}} = \sqrt{1+m^2}\,|y_1 - w_1|\,.$$

Therefore, equation (5.1) holds. □

Lets apply the content of Lemma 5.1 in the Lorentzian plane. We use the distance (2.8) of the Lorentzian plane.

**Lemma 5.2.** *For any collinear points $X$, $Y$, $Z$, $W$ in the Lorentzian plane are given in that order along their line,*

$$XW_{\mathbb{L}} \cdot YZ_{\mathbb{L}} + XY_{\mathbb{L}} \cdot WZ_{\mathbb{L}} = XZ_{\mathbb{L}} \cdot WY_{\mathbb{L}}. \tag{5.2}$$

*Proof.* Let $y = mx + n$ be the straight line passing through the points $X$, $Y$, $Z$, and $W$. For $x_1 < y_1 < z_1 < w_1$, let point $X$ be $X(x_1, mx_1+n)$, point $Y$ be $Y(y_1, my_1+n)$, point $Z$ be $Z(z_1, mz_1 + n)$, and point $W$ be $W(w_1, mw_1 + n)$. Applying (2.8) to calculate the distance between two points gives us

$$XW_{\mathbb{L}} = \sqrt{1-m^2}\,|w_1 - x_1|\,, \qquad YZ_{\mathbb{L}} = \sqrt{1-m^2}\,|z_1 - y_1|\,,$$
$$XY_{\mathbb{L}} = \sqrt{1-m^2}\,|y_1 - x_1|\,, \qquad WZ_{\mathbb{L}} = \sqrt{1-m^2}\,|z_1 - w_1|\,,$$

and

$$XZ_{\mathbb{L}} = \sqrt{1-m^2}\,|z_1 - x_1|\,, \qquad WY_{\mathbb{L}} = \sqrt{1-m^2}\,|y_1 - w_1|\,.$$

Therefore, equation (5.2) holds. □

Finally, let us apply Lemma 5.1 to the isotropic plane. We use the distance (2.9) in the isotropic plane.

**Lemma 5.3.** *For any collinear points $X$, $Y$, $Z$, $W$ in the isotropic plane are given in that order along their line,*

$$XW_{\mathbb{I}} \cdot YZ_{\mathbb{I}} + XY_{\mathbb{I}} \cdot WZ_{\mathbb{I}} = XZ_{\mathbb{I}} \cdot WY_{\mathbb{I}}. \tag{5.3}$$

*Proof.* Let $y = mx + n$ be the straight line passing through the points $X$, $Y$, $Z$, and $W$. For $x_1 < y_1 < z_1 < w_1$, let point $X$ be $X(x_1, mx_1+n)$, point $Y$ be $Y(y_1, my_1+n)$, point $Z$ be $Z(z_1, mz_1 + n)$, and point $W$ be $W(w_1, mw_1 + n)$. Applying (2.9) to calculate the distance between two points gives us

$$XW_{\mathbb{I}} = |w_1 - x_1|\,, \qquad YZ_{\mathbb{I}} = |z_1 - y_1|\,,$$
$$XY_{\mathbb{I}} = |y_1 - x_1|\,, \qquad WZ_{\mathbb{I}} = |z_1 - w_1|\,,$$

and

$$XZ_{\mathbb{I}} = |z_1 - x_1|, \qquad WY_{\mathbb{I}} = |y_1 - w_1|.$$

Therefore, equation (5.3) holds. □

In the Euclidean plane, Lorentzian plane, and isotropic plane,

$$XW \cdot YZ + XY \cdot WZ = XZ \cdot WY$$

holds.

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